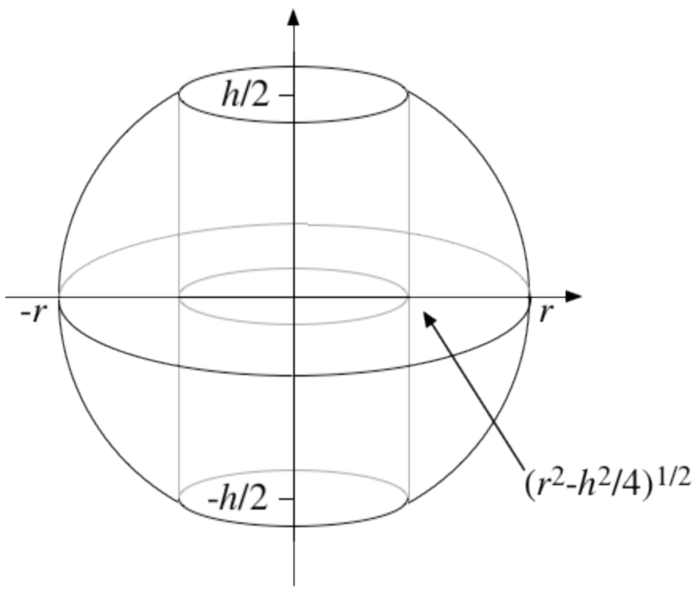
Mathematics 1100Y – Calculus I: Calculus of one variable
TRENT UNIVERSITY, SUMMER 2011

Solution to Assignment #9
The Hole Thing

A cylindrical hole is made through a sphere of radius r, after which the resulting solid has height h.



1. Find the volume of this solid. *[10]*

HINT: Surprisingly, the answer does not depend on r, except for the obvious requirement that $h \leq 2r$.

SOLUTION. This solid can be obtained by rotating the region between the circle $x^2+y^2 = r^2$ and the line $x = a$, with a chosen so that $a^2 + (h/2)^2 = r^2$, about the y-axis. (See the modified sketch above.) We can solve for the necessary a in terms of r and h:

$$a^2 + (h/2)^2 = r^2 \quad \Longrightarrow \quad a^2 = r^2 - (h/2)^2 = r^2 - h^2/4 \quad \Longrightarrow \quad a = \sqrt{r^2 - h^2/4}$$

We will find the volume of this solid of revolution using the washer method. Since the region was rotated about the y-axis, we need to integrate with respect to y; note that the limits for y will be $-h/2$ and $h/2$. To avoid confusion with the r we already have in the problem, namely the radius of the sphere, we will use S for the outside radius of the washer at y and s for the inside radius. Then $S = x$ for the x such that $x^2 + y^2 = r^2$, so $S = \sqrt{r^2 - y^2}$, and $s = a = \sqrt{r^2 - h^2/4}$. Plugging all this into the volume formula for washers gives:

$$
\begin{aligned}
\text{Volume} &= \int_{-h/2}^{h/2} \pi \left[S^2 - s^2\right]\, dy \\
&= \pi \int_{-h/2}^{h/2} \left[\left(\sqrt{r^2 - y^2}\right)^2 - \left(\sqrt{r^2 - h^2/4}\right)^2\right]\, dy \\
&= \pi \int_{-h/2}^{h/2} \left[\left(r^2 - y^2\right) - \left(r^2 - h^2/4\right)\right]\, dy \\
&= \pi \int_{-h/2}^{h/2} \left[h^2/4 - y^2\right]\, dy \qquad \text{It should now be apparent that the answer will not involve } r \ldots \\
&= \pi \int_{-h/2}^{h/2} \frac{h^2}{4}\, dy - \pi \int_{-h/2}^{h/2} y^2\, dy \\
&= \pi\, \frac{h^2}{4} y\Big|_{-h/2}^{h/2} - \pi\, \frac{y^3}{3}\Big|_{-h/2}^{h/2} \\
&= \pi \frac{h^2}{4}\left[\frac{h}{2} - \left(-\frac{h}{2}\right)\right] - \frac{\pi}{3}\left[\left(\frac{h}{2}\right)^3 - \left(-\frac{h}{2}\right)^3\right] \\
&= \pi \frac{h^2}{4} \cdot 2\frac{h}{2} - \frac{\pi}{3} \cdot 2\frac{h^3}{8} \\
&= \frac{\pi}{4} h^3 - \frac{\pi}{12} h^3 \\
&= \frac{2\pi}{12} h^3 = \frac{\pi}{6} h^3 \quad \square
\end{aligned}
$$

Alternate Solution. If you know that the value of r doesn't matter, as the hint tells us, you can just pick one. The most convenient one is the smallest one possible, namely $r = h/2$. A sphere of radius $r = h/2$ will have height h; in this case the cylindrical hole has to have width 0, so it takes away nothing from the volume of the sphere. Since a sphere of radius r has volume $\frac{4}{3}\pi r^3$, it follows that the sphere of radius $r = h/2$ has volume $\frac{4}{3}\pi\left(\frac{h}{2}\right)^3 = \frac{4}{3}\pi \cdot \frac{h^3}{2^3} = \frac{4}{3} \cdot \frac{\pi}{8} h^3 = \frac{\pi}{6} h^3$. As the value of r doesn't really matter, any solid of the sort considered in the original question has this volume. $\square$

Note: A version of this problem occurs in the textbook, §6.3 #46, and another, §6.2 #70, is closely related. The version given in this assignment was adapted slightly from one Martin Gardner's columns on recreational mathematics in *Scientific American.* (Gardner, in turn, apparently got it from a periodical called *The Graham Dial,* and traced it back to a book called *Mathematical Nuts* by Samuel I. Jones.) When it appeared in Gardner's column, without the hint that that appears on the assignment, something very close to the alternate solution above was given by the science-fiction editor John W. Campbell, Jr.