Mathematics 1100Y – Calculus I: Calculus of one variable
TRENT UNIVERSITY, Summer 2011

Solutions to the Quizzes

Quiz #1. Wednesday, 11 May, 2011. *[10 minutes]*

1. Compute $\lim_{x \to -3} \dfrac{x+3}{x^2-9}$ using the appropriate limit laws and algebra. *[5]*

SOLUTION. Note that $\lim_{x \to -3}(x+3) = -3+3 = 0$ and $\lim_{x \to -3}(x^2-9) = (-3)^2-9 = 9-9 = 0$, so we can't use the limit law for quotients lest we divide by 0. We will resort to algebra instead, using the fact that $x^2-9 = (x+3)(x-3)$. Then

$$\begin{aligned}\lim_{x \to -3} \frac{x+3}{x^2-9} &= \lim_{x \to -3} \frac{x+3}{(x+3)(x-3)} = \lim_{x \to -3} \frac{1}{x-3} \\ &= \frac{1}{\lim_{x \to -3}(x-3)} = \frac{1}{-3-3} = \frac{1}{-6} = -\frac{1}{6}\,. \quad \square\end{aligned}$$

Quiz #2. Monday, 16 May, 2011. *[10 minutes]*

Do *one* of questions 1 or 2.

1. Use the ε-δ definition of limits to verify that $\lim_{x \to 1}(3x-2) = 1$. *[5]*

SOLUTION. We need to check that for any $\varepsilon > 0$ there is a $\delta > 0$ such that if $|x-1| < \delta$, then $|(3x-2)-1| < \varepsilon$. As usual, we will try to reverse-engineer the required δ. Suppose $\varepsilon > 0$ is given. Then

$$\begin{aligned}|(3x-2)-1| < \varepsilon &\iff |3x-3| < \varepsilon \\ &\iff |3(x-1)| < \varepsilon \\ &\iff |x-1| < \frac{\varepsilon}{3}\,.\end{aligned}$$

Since very step above is reversible, if $|x-1| < \frac{\varepsilon}{3}$, then $|(3x-2)-1| < \varepsilon$. Hence $\delta = \frac{\varepsilon}{3}$ wil do the job, and so, by the ε-δ definition of limits, $\lim_{x \to 1}(3x-2) = 1$. $\square$

2. Find the x- and y-intercepts and all the horizontal asymptotes of $f(x) = \dfrac{x^2}{x^2+1}$, and sketch its graph. *[5]*

SOLUTION. *i. (Intercepts)* Since $f(0) = \dfrac{0^2}{0^2+1} = \dfrac{0}{1} = 0$, the y-intercept is the point $(0,0)$. $f(x) = \dfrac{x^2}{x^2+1} = 0$ is only possible when $x = 0$, so the point $(0,0)$ is also the only x-intercept.

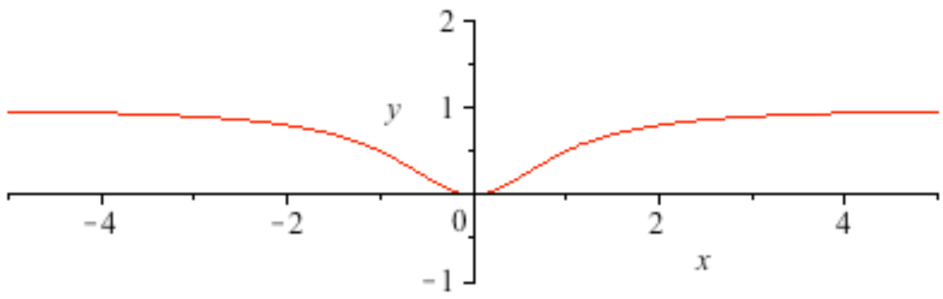
ii. (Horizontal asymptotes) We check the limits of $f(x)$ as $x \to \pm\infty$:

$$\lim_{x\to+\infty} \frac{x^2}{x^2+1} = \lim_{x\to+\infty} \frac{x^2}{x^2+1} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}} = \lim_{x\to+\infty} \frac{\frac{x^2}{x^2}}{\frac{x^2}{x^2}+\frac{1}{x^2}} = \lim_{x\to+\infty} \frac{1}{1+\frac{1}{x^2}} = \frac{1}{1+0} = 1$$

$$\lim_{x\to-\infty} \frac{x^2}{x^2+1} = \lim_{x\to-\infty} \frac{x^2}{x^2+1} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}} = \lim_{x\to-\infty} \frac{\frac{x^2}{x^2}}{\frac{x^2}{x^2}+\frac{1}{x^2}} = \lim_{x\to-\infty} \frac{1}{1+\frac{1}{x^2}} = \frac{1}{1+0} = 1$$

(Note that $\frac{1}{x^2} \to 0$ as $x \to \pm\infty$.) It follows that $f(x)$ has $y = 1$ as a horizontal asymptote in both directions.

iii. (Graph) Note that since $x^2 < x^2+1$, $0 \le \dfrac{x^2}{x^2+1} < 1$. Together with the information obtained above, this means that the graph of $f(x)$ looks like:



I cheated a bit to draw this, of course; this graph was made by `Maple` with the command:

```
> plot(x^2/(x^2+1), x=-5..5, y=-1..2);
```

The ">" is `Maple`'s prompt. □

Quiz #3. Wednesday, 18 May, 2011. *[10 minutes]*

1. Use the limit definition of the derivative to compute $f'(a)$ for $f(x) = \dfrac{1}{x}$. (You may assume that $a \neq 0$.) *[5]*

Solution. By the limit definition of the derivative,

$$\begin{aligned} f'(a) &= \lim_{h\to0} \frac{f(a+h)-f(a)}{h} = \lim_{h\to0} \frac{\frac{1}{a+h}-\frac{1}{a}}{h} = \lim_{h\to0} \frac{1}{h} \cdot \frac{a-(a+h)}{(a+h)a} \\ &= \lim_{h\to0} \frac{1}{h} \cdot \frac{-h}{(a+h)a} = \lim_{h\to0} \frac{-1}{(a+h)a} = \frac{-1}{(a+0)a} = \frac{-1}{a^2}. \qquad \square \end{aligned}$$

Quiz #4. Wednesday, 25 May, 2011. *[10 minutes]*

1. Compute $f'(x)$ for $f(x) = \ln\left(\dfrac{x}{1+x^2}\right)$. *[5]*

Solution. Using the Chain Rule and Quotient Rule at the key steps, and simplifying as much as can easily be done at the end:

$$\begin{aligned}
f'(x) &= \frac{d}{dx}\left(\ln\left(\frac{x}{1+x^2}\right)\right)\\
&= \frac{1}{\frac{x}{1+x^2}} \cdot \frac{d}{dx}\left(\frac{x}{1+x^2}\right)\\
&= \frac{1+x^2}{x} \cdot \frac{\frac{dx}{dx}\cdot\left(1+x^2\right) - x \cdot \frac{d}{dx}\left(1+x^2\right)}{\left(1+x^2\right)^2}\\
&= \frac{1}{x} \cdot \frac{1\cdot\left(1+x^2\right) - x\cdot(0+2x)}{1+x^2}\\
&= \frac{1}{x} \cdot \frac{1+x^2-2x^2}{1+x^2}\\
&= \frac{1}{x} \cdot \frac{1-x^2}{1+x^2}\\
&= \frac{1-x^2}{x\left(1+x^2\right)} \qquad \square
\end{aligned}$$

Quiz #5. Monday, 30 May, 2011. *[10 minutes]*

Do *one* of questions 1 or 2.

1. Find $\dfrac{dy}{dx}$ at the point $(2,2)$ on the curve defined by $x = \sqrt{x+y}$. *[5]*

Solution i. We differentiate both sides of the equation defining the curve,

$$1 = \frac{dx}{dx} = \frac{d}{dx}\sqrt{x+y} = \frac{1}{2\sqrt{x+y}} \cdot \frac{d}{dx}(x+y) = \frac{1}{2\sqrt{x+y}} \cdot \left(\frac{dx}{dx} + \frac{dy}{dx}\right) = \frac{1+\frac{dy}{dx}}{2\sqrt{x+y}},$$

and then solve for $\dfrac{dy}{dx}$,

$$\frac{1+\frac{dy}{dx}}{2\sqrt{x+y}} = 1 \quad\Longrightarrow\quad 1+\frac{dy}{dx} = 2\sqrt{x+y} \quad\Longrightarrow\quad \frac{dy}{dx} = 2\sqrt{x+y}-1\,.$$

It follows that $\left.\dfrac{dy}{dx}\right|_{(x,y)=(2,2)} = 2\sqrt{2+2}-1 = 2\sqrt{4}-1 = 2\cdot 2 - 1 = 4-1 = 3.$ $\square$

Solution ii. We solve for y first,

$$x = \sqrt{x+y} \quad\Longrightarrow\quad x^2 = x+y \quad\Longrightarrow\quad y = x^2 - x\,,$$

and then differentiate, $\dfrac{dy}{dx} = \dfrac{d}{dx}\left(x^2 - x\right) = 2x - 1$. It follows that $\left.\dfrac{dy}{dx}\right|_{(x,y)=(2,2)} = 2 \cdot 2 - 1 = 4 - 1 = 3$. $\square$

2. Find $\dfrac{dy}{dx}$ at $x = e$ for $y = \ln\left(x\ln(x)\right)$. *[5]*

SOLUTION I. We differentiate directly, using the Chain and Product Rules:

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\ln\left(x\ln(x)\right) = \frac{1}{x\ln(x)} \cdot \frac{d}{dx}\left(x\ln(x)\right) = \frac{1}{x\ln(x)} \cdot \left(\left[\frac{dx}{dx}\right] \cdot \ln(x) + x \cdot \left[\frac{d}{dx}\ln(x)\right]\right) \\ &= \frac{1}{x\ln(x)} \cdot \left(1 \cdot \ln(x) + x \cdot \frac{1}{x}\right) = \frac{\ln(x)+1}{x\ln(x)}\end{aligned}$$

It follows that $\left.\dfrac{dy}{dx}\right|_{x=e} = \dfrac{\ln(e)+1}{e\ln(e)} = \dfrac{1+1}{e \cdot 1} = \dfrac{2}{e}$. $\square$

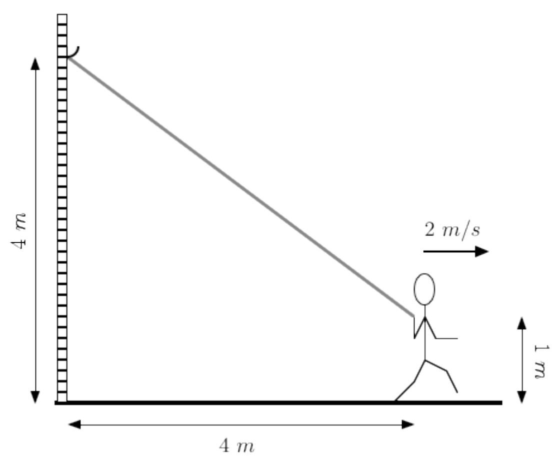
SOLUTION II. We rewrite y using the properties of logarithms, $y = \ln\left(x\ln(x)\right) = \ln(x) + \ln\left(\ln(x)\right)$, and then differentiate, using the Sum and Chain Rules:

$$\frac{dy}{dx} = \frac{d}{dx}\left(\ln(x) + \ln\left(\ln(x)\right)\right) = \frac{1}{x} + \frac{1}{\ln(x)} \cdot \frac{d}{dx}\ln(x) = \frac{1}{x} + \frac{1}{\ln(x)} \cdot \frac{1}{x} = \frac{1}{x}\left(1 + \frac{1}{\ln(x)}\right)$$

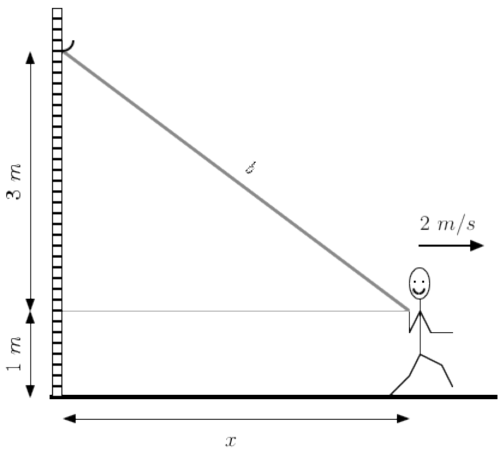
It follows that $\left.\dfrac{dy}{dx}\right|_{x=e} = \dfrac{1}{e}\left(1 + \dfrac{1}{\ln(e)}\right) = \dfrac{1}{e}\left(1 + \dfrac{1}{1}\right) = \dfrac{2}{e}$. $\square$

Quiz #6. Wednesday, 1 June, 2011. *[15 minutes]*

1. A $3\,m$ long, very stretchy, bungee cord is suspended from a hook $4\,m$ up on a wall. The other end of the cord is grabbed by a child who runs directly away from the wall at $2\,m/s$, holding the end of the cord $1\,m$ off the ground, stretching the cord in the process. How is the length of the cord changing at the instant that the child's end of the cord is $4\,m$ away from the wall? *[5]*



SOLUTION. Call the distance the child is from the wall x and observe that the bungee cord forms the hypotenuse of a right triangle with base x and height $4 - 1 = 3$. (Recall that the child holds the end of the cord $1\,m$ above the ground, while the top end of the cord is attached to the wall $4\,m$ above the ground.) The fact that the child is running at $2\,m/s$ amounts to saying that $\frac{dx}{dt} = 2$.



If we denote the length of the bungee cord by b, then we have $x^2 + 3^2 = b^2$ by the Pythagorean theorem, and are trying to discover what $\frac{db}{dt}$ is when $x = 4$.

We first differentiate both sides of the equation,

$$x^2 + 3^2 = b^2 \quad \Longrightarrow \quad \frac{d}{dt}\left(x^2 + 3^2\right) = \frac{d}{dt}b^2 \quad \Longrightarrow \quad 2x\frac{dx}{dt} + 0 = 2b\frac{db}{dt}\,,$$

and solve for $\frac{db}{dt}$,

$$\frac{db}{dt} = \frac{2x}{2b} \cdot \frac{dx}{dt} = \frac{x}{b} \cdot \frac{dx}{dt}\,.$$

We know that $\frac{dx}{dt} = 2$ and that $x = 4$ at the instant in question. It follows from $x^2 + 3^2 = b^2$ that $b = \sqrt{x^2 + 3^2}$, so $b = \sqrt{4^2 + 3^2} = \sqrt{16 + 9} = \sqrt{25} = 5$ when $x = 4$. Thus, when $x = 4$, $\frac{db}{dt} = \frac{x}{b} \cdot \frac{dx}{dt} = \frac{4}{5} \cdot 2 = \frac{8}{5} = 1.6\,m/s$. $\square$

Quiz #7. Monday, 6 June, 2011. *[15 minutes]*

1. Find any and all intercepts, intervals of increase and decrease, local maxima and minima, and vertical and horizontal asymptotes, of $y = xe^{-x}$, and sketch this curve based on the information you obtained. *[5]*

Bonus: Find any and all the points of inflection of this curve too. *[1]*

Hint: You may assume that $\lim_{x \to +\infty} xe^{-x} = 0$. For $\lim_{x \to -\infty} xe^{-x}$ you're on your own.

SOLUTION. Here goes!

i. *(Domain)* The expression xe^{-x} is defined for all x. Since it is a product of continuous functions, it is continuous too. It follows, in particular, that there are no vertical asymptotes.

ii. *(Intercepts)* For the y-intercept we plug in $x = 0$, then $y = 0e^{-0} = 0 \cdot 1 = 0$. For the x-intercept, note that since $e^{-x} > 0$ for all x, $xe^{-x} = 0$ exactly when $x = 0$. Thus the origin, $(0, 0)$ is both the y- and the only x-intercept.

iii. *(Vertical asymptotes)* None, for the reasons noted in *i* above.

iv. *(Horizontal asymptotes)* The hint tells us that we can assume that $\lim_{x \to +\infty} xe^{-x} = 0$, so curve has the horizontal asymptote $y = 0$ in the +ve direction. On the other hand, since $e^{-x} \to +\infty$ as $x \to -\infty$, $\lim_{x \to -\infty} xe^{-x} = -\infty$, there is no horizontal asymptote in the −ve direction.

v. *(Increase, decrease, maxima, & minima)* First, we differentiate, using the Product and Chain Rules:

$$\frac{dy}{dx} = \frac{d}{dx}\left(xe^{-x}\right) = \left(\frac{d}{dx}x\right) \cdot e^{-x} + x \cdot \left(\frac{d}{dx}e^{-x}\right) = 1 \cdot e^{-x} + x \cdot e^{-x} \cdot \frac{d}{dx}(-x)$$
$$= e^{-x} + xe^{-x}(-1) = (1 - x)e^{-x}$$

Since $e^{-x} > 0$ for all x, it follows that:

$$\frac{dy}{dx} = (1-x)e^{-x} \begin{array}{c} > \\ = \\ < \end{array} 0 \iff (1-x) \begin{array}{c} > \\ = \\ < \end{array} 0 \iff x \begin{array}{c} < \\ = \\ > \end{array} 1$$

Building the usual table:

x	$(-\infty, 1)$	1	$(1, \infty)$
y	$\uparrow$	max	$\downarrow$
$\frac{dy}{dx}$	$+$	0	$-$

In words: $y = xe^{-x}$ is increasing on $(\infty, 1)$, has a maximum at 1, and is decreasing on $(1, +\infty)$. Note that since $y = xe^{-x}$ is defined and continuous everywhere, the maximum is an absolute maximum. From our work looking horizontal asymptotes in *iv* above, it is clear that there is no absolute minimum.

vi. *(Bonus – Points of inflection)* We differentiate $\frac{dy}{dx}$, using the Product and Chain Rules, to get $\frac{d^2y}{dx^2}$:

$$\begin{aligned}\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left((1-x)e^{-x}\right) &= \left(\frac{d}{dx}(1-x)\right)\cdot e^{-x} + (1-x)\cdot\left(\frac{d}{dx}e^{-x}\right) \\ &= (-1)\cdot e^{-x} + (1-x)\cdot e^{-x}\cdot\frac{d}{dx}(-x) \\ &= -e^{-x} + (1-x)e^{-x}(-1) = (x-2)e^{-x}\end{aligned}$$

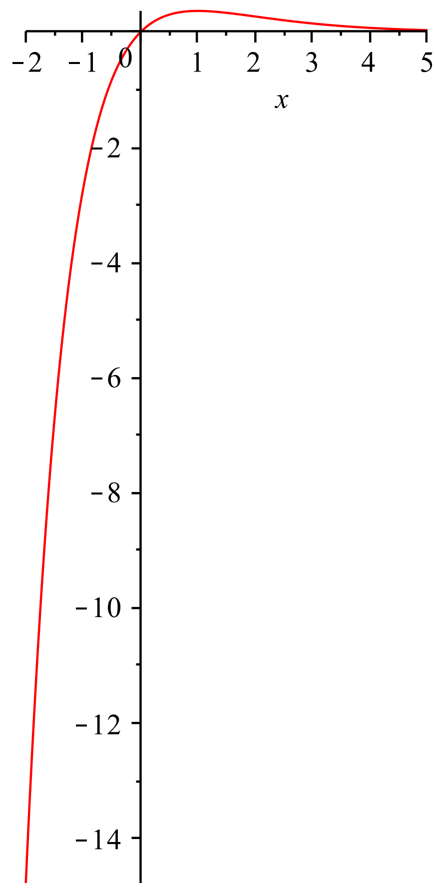
Since $e^{-x} > 0$ for all x, it follows that:

$$\frac{d^2y}{dx^2} = (x-2)e^{-x} \begin{array}{c}>\\=\\<\end{array} 0 \iff (x-2) \begin{array}{c}>\\=\\<\end{array} 0 \iff x \begin{array}{c}>\\=\\<\end{array} 2$$

Thus $y = xe^{-x}$ is concave down on $(-\infty, 2)$ and concave up on $(2, +\infty)$, so $x = 2$ gives the only point of inflection.

vii. *(Graph)* The following was drawn using `Maple` with the command

```
> plot(x*exp(-x),x=-2..5);
```



That's all, folks! □

Quiz #8. Monday, 13 June, 2011. *[10 minutes]*

1. Compute $\displaystyle\lim_{x\to\infty} \frac{x^2}{e^x}$. *[5]*

SOLUTION. Since $x^2 \to \infty$ and $e^x \to \infty$ as $x \to \infty$, we can apply l'Hôpital's Rule.

$$\lim_{x\to\infty} \frac{x^2}{e^x} = \lim_{x\to\infty} \frac{\frac{d}{dx}x^2}{\frac{d}{dx}e^x} = \lim_{x\to\infty} \frac{2x}{e^x}$$

At this point, note that $2x \to \infty$ and $e^x \to \infty$ as $x \to \infty$, so we can apply l'Hôpital's Rule again. Thus

$$\lim_{x\to\infty} \frac{x^2}{e^x} = \lim_{x\to\infty} \frac{2x}{e^x} = \lim_{x\to\infty} \frac{\frac{d}{dx}(2x)}{\frac{d}{dx}e^x} = \lim_{x\to\infty} \frac{2}{e^x} = 0,$$

since $2 \to 2$ and $e^x \to \infty$ as $x \to \infty$. □

Quiz #9. Monday, 20 June, 2011. *[10 minutes]*

Do *one* of questions 1, 2, or 3.

1. Compute $\int_1^2 (x+1)\,dx$ using the Right-Hand Rule. *[5]*

   Hint: You may assume that $1 + 2 + 3 + \cdots + n = \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$.

SOLUTION. We'll throw the Right-Hand Rule formula,

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{b-a}{n} f\left(a + i\frac{b-a}{n}\right),$$

at this problem and hope we survive the algebra! Note that $f(x) = x + 1$ in this case.

$$\begin{aligned}
\int_1^2 (x+1)\,dx &= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{2-1}{n} f\left(1 + i\frac{2-1}{n}\right) = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n} f\left(1 + \frac{i}{n}\right) \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n}\left(\left(1 + \frac{i}{n}\right) + 1\right) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \left(2 + \frac{i}{n}\right) \\
&= \lim_{n\to\infty} \frac{1}{n}\left[\left(\sum_{i=1}^{n} 2\right) + \left(\sum_{i=1}^{n} \frac{i}{n}\right)\right] = \lim_{n\to\infty} \frac{1}{n}\left[2n + \frac{1}{n}\left(\sum_{i=1}^{n} i\right)\right] \\
&= \lim_{n\to\infty} \frac{1}{n}\left[2n + \frac{1}{n} \cdot \frac{n(n+1)}{2}\right] = \lim_{n\to\infty} \frac{1}{n}\left[2n + \frac{n+1}{2}\right] \\
&= \lim_{n\to\infty} \frac{1}{n}\left[\frac{5}{2}n + \frac{1}{2}\right] = \lim_{n\to\infty}\left[\frac{5}{2} + \frac{1}{2n}\right] = \frac{5}{2} + 0 = \frac{5}{2},
\end{aligned}$$

since $\frac{1}{2n} \to 0$ as $n \to \infty$. □

2. Compute $\int_{-1}^{3} (x+1)^2\, dx$. *[5]*

SOLUTION. We'll expand $(x+1)^2$ and then use the Power, Sum, and Constant Rules (all at once!) to find the antiderivative and (implicitly!) apply the Fundamental Theorem of Calculus to evaluate the definite integral.

$$\begin{aligned}
\int_{-1}^{3} (x+1)^2\, dx &= \int_{-1}^{3} \left(x^2+2x+1\right)\, dx = \left(\frac{x^3}{3} + 2\frac{x^2}{2} + x\right)\Big|_{-1}^{3} \\
&= \left(\frac{3^3}{3} + 3^2 + 3\right) - \left(\frac{(-1)^3}{3} + (-1)^2 + (-1)\right) \\
&= (9+9+3) - \left(-\frac{1}{3} + 1 - 1\right) = 21 - \left(-\frac{1}{3}\right) \\
&= \frac{63}{3} + \frac{1}{3} = \frac{64}{3}
\end{aligned}$$

Note that because we were finding the antiderivative to evaluate an indefinite integral, we did not worry about adding a generic constant of C to the antiderivative. (If we had, it would have cancelled out when we evaluated the definite integral anyway.) $\square$

3. Compute $\int \sin(x)\cos(x)\, dx$. *[5]*

SOLUTION. This is a job for the Substitution and Power Rules. We will use the substitution $u = \sin(x)$, so $\dfrac{du}{dx} = \sin(x)$ and $du = \sin(x)\, dx$:

$$\int \sin(x)\cos(x)\, dx = \int u\, du = \frac{u^2}{2} + C = \frac{\sin^2(x)}{2} + C$$

Note that since we were computing an indefinite integral, we need to find the most general possible antiderivative and so must add the generic constant C. $\square$

Quiz #10. Wednesday, 22 June, 2011. *[10 minutes]*

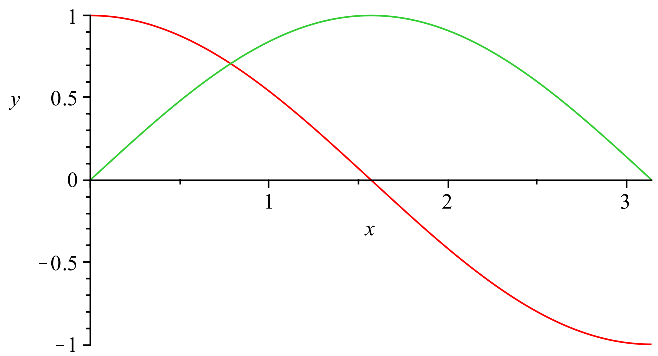
1. Find the area of the region between the curves $y = \cos(x)$ and $y = \sin(x)$, where $0 \le x \le \pi$. *[5]*

   Hint: Recall that $\sin\left(\frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$.

SOLUTION. From $x = 0$ to $x = \pi$, $\cos(x)$ starts at 1, decreases to 0 at $x = \frac{\pi}{2}$, and decreases some more to -1, while $\sin(x)$ starts at 0, increases to 1 at $x = \frac{\pi}{2}$, and then decreases to 0. The two graphs cross once in the interval $[0, \pi]$: $\cos(x) = \sin(x)$ at $x = \frac{\pi}{4}$. *(Hint!)* It follows that $\cos(x) \ge \sin(x)$ for $0 \le x \le \frac{\pi}{4}$, and $\sin(x) \ge \cos(x)$ for $\frac{pi}{4} \le x \le \pi$.

It is a little easier to see all this if you can visualize or draw the graphs. Here is a graph produced by `Maple`:

```
> plot([[t,cos(t),t=0..Pi],[t,sin(t),t=0..Pi]],x=0..Pi,y=-1..1);
```



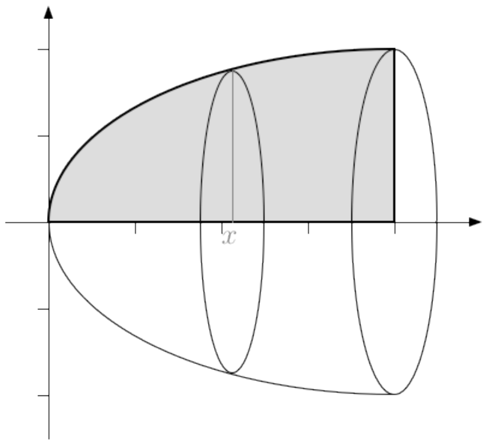
We will break up the integral according to our analysis above:

$$\begin{aligned}
\text{Area} &= \int_0^{\pi/4} [\cos(x) - \sin(x)]\ dx + \int_{\pi/4}^{\pi} [\sin(x) - \cos(x)]\ dx \\
&= [\sin(x) - (-\cos(x))]\Big|_0^{\pi/4} + [(-\cos(x)) - \sin(x)]\Big|_{\pi/4}^{\pi} \\
&= [\sin(x) + \cos(x)]\Big|_0^{\pi/4} + [-\cos(x) - \sin(x)]\Big|_{\pi/4}^{\pi} \\
&= \left[\sin\left(\frac{\pi}{4}\right) + \cos\left(\frac{\pi}{4}\right)\right] - [\sin(0) + \cos(0)] \\
&\qquad + [-\cos(\pi) - \sin(\pi)] - \left[-\cos\left(\frac{\pi}{4}\right) - \sin\left(\frac{\pi}{4}\right)\right] \\
&= \left[\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right] - [0 + 1] + [-(-1) - 0] - \left[-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right] \\
&= \frac{2}{\sqrt{2}} - 1 + 1 - \left[-\frac{2}{\sqrt{2}}\right] = \frac{4}{\sqrt{2}} = 2\sqrt{2} \qquad \square
\end{aligned}$$

Quiz #11. Monday, 27 June, 2011. *[10 minutes]*

1. Sketch the solid obtained by rotating the region bounded by $y = \sqrt{x}$ and $y = 0$, where $0 \le x \le 4$, about the x-axis and find its volume. *[5]*

SOLUTION. Here's a sketch of the solid, with the original region shaded in:



We will use the washer method to find the volume of the solid. Since we are using washers and rotated about a horizontal line, we will be integrating with respect to x, the variable belonging to the horizontal axis. The cross section of the solid at x has outside radius $R = y = \sqrt{x}$ and inside radius $r = 0$ (so it is actually a disk). Plugging these into the volume formula for the washer method gives:

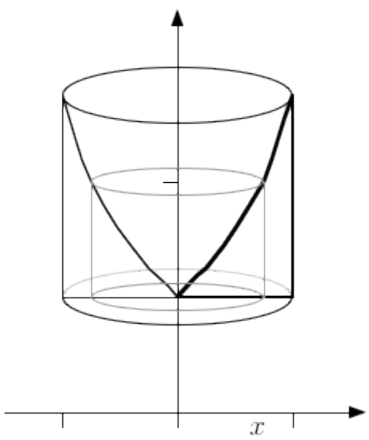
$$\begin{aligned}\text{Volume} &= \int_0^4 \pi \left(R^2 - r^2\right)\, dx = \pi \int_0^4 \left(\left(\sqrt{x}\right)^2 - 0^2\right)\, dx = \pi \int_0^4 (x - 0)\, dx \\ &= \pi \int_0^4 x\, dx = \pi \left.\frac{x^2}{2}\right|_0^4 = \pi \left(\frac{4^2}{2} - \frac{0^2}{2}\right) = \pi \left(\frac{16}{2} - 0\right) = 8\pi\end{aligned}$$

Note that the limits for the integral, as always, come from the original region. $\square$

Quiz #12. Wednesday, 29 June, 2011. *[10 minutes]*

1. Sketch the solid obtained by rotating the region between $y = e^x$ and $y = 1$, where $0 \leq x \leq 1$, about the y-axis and find its volume. *[5]*

SOLUTION. Here's a sketch of the solid, with the original region outlined in a bolder line and with a typical cylindrical shell drawn in as well:



We will use the method of cylindrical shells to find the volume of the solid. Since we are using shells and rotated about a vertical line (the y-axis, otherwise known as $x = 0$), we will be integrating with respect to x, the variable belonging to the horizontal axis. The cylindrical shell at x has radius $r = x - 0 = x$ and height $h = e^x - 1$. Plugging these into the volume formula for the shell method gives:

$$\begin{aligned}
\text{Volume} &= \int_0^1 2\pi r h \, dx = \int_0^1 2\pi x \left(e^x - 1\right) dx = 2\pi \int_0^1 \left(xe^x - x\right) dx \\
&= 2\pi \int_0^1 xe^x \, dx - 2\pi \int_0^1 x \, dx \qquad \text{Use integration by parts on the} \\
&\qquad \text{first piece, with } u = x \text{ and } v' = e^x \text{, so } u' = 1 \text{ and } v = e^x. \\
&\qquad \text{(Recall the integration by parts formula: } \int uv' \, dx = uv - \int u'v \, dx.\text{)} \\
&= 2\pi \left[xe^x \Big|_0^1 - \int_0^1 1e^x \, dx \right] - 2\pi \left. \frac{x^2}{2} \right|_0^1 = 2\pi \left[1e^e - 0e^0 - e^x\Big|_0^1 \right] - 2\pi \left[\frac{1^2}{2} - \frac{0^2}{2} \right] \\
&= 2\pi \left[e - 0 - \left(e^1 - e^0\right) \right] - 2\pi \left[\frac{1}{2} - 0 \right] = 2\pi[e - e + 1] - 2\pi \frac{1}{2} = 2\pi - \pi = \pi
\end{aligned}$$

Note that the limits for the integral, as always, come from the original region. □

Quiz #13. Monday, 4 July, 2011. *[10 minutes]*

1. Compute $\displaystyle\int \sec^3(x)\tan^3(x)\,dx$. *[5]*

SOLUTION. We'll use a combination of the trig identity $\tan^2(x) = \sec^2(x) - 1$ and the fact that $\dfrac{d}{dx}\sec(x) = \sec(x)\tan(x)$ to set up a suitable substitution:

$$\begin{aligned}
\int \sec^3(x)\tan^3(x)\,dx &= \int \sec^2(x)\tan^2(x)\sec(x)\tan(x)\,dx \\
&= \int \sec^2(x)\left(\sec^2(x) - 1\right)\sec(x)\tan(x)\,dx \\
&\quad \text{Substitute } u = \sec(x)\text{, so } du = \sec(x)\tan(x)\,dx. \\
&= \int u^2\left(u^2 - 1\right)\,du = \int \left(u^4 - u^2\right)\,du \\
&= \frac{u^5}{5} - \frac{u^3}{5} + C = \frac{\sec^5(x)}{5} - \frac{\sec^3(x)}{5} + C \qquad \square
\end{aligned}$$

Quiz #14. Monday, 11 July, 2011. *[15 minutes]*

1. Compute $\displaystyle\int \frac{1}{x^4 + x^2}\,dx$. *[5]*

SOLUTION. Since the integrand is a rational function, we will compute the integral using partial fractions. The first thing that we need to do is to factor the denominator. It is easy to see that $x^4 + x^2 = \left(x^2 + 1\right)x^2$.

Since x^2+1 is irreducible (as $x^2+1 \geq 1 > 0$ for all x, it has no roots) and $x^2 = (x-0)^2$ is a repeated linear factor, the partial fraction decomposition of the integrand will have the form:

$$\frac{1}{x^4 + x^2} = \frac{1}{(x^2+1)\,x^2} = \frac{Ax + B}{x^2 + 1} + \frac{C}{x^2} + \frac{D}{x}$$

Next, we need to work out what the constants A, B, C, and D are. Combining the right-hand side of the above equation over the common denominator of $\left(x^2+1\right)x^2$ and collecting like terms in the numerator gives:

$$\begin{aligned}
\frac{1}{(x^2+1)\,x^2} &= \frac{Ax+B}{x^2+1} + \frac{C}{x^2} + \frac{D}{x} = \frac{(Ax+B)x^2 + C\left(x^2+1\right) + D\left(x^2+1\right)x}{(x^2+1)\,x^2} \\
&= \frac{Ax^3 + Bx^2 + Cx^2 + C + Dx^3 + Dx}{(x^2+1)\,x^2} = \frac{(A+D)x^3 + (B+C)x^2 + Dx + C}{(x^2+1)\,x^2}
\end{aligned}$$

Comparing numerators, we have $1 = (A+D)x^3 + (B+C)x^2 + Dx + C$, so we must have $A + D = 0$, $B + C = 0$, $D = 0$, and $C = 1$. It follows that $A + 0 = 0$, so $A = 0$, and $B + 1 = 0$, so $B = -1$. Thus the partial fraction decomposition of the given integrand is:

$$\frac{1}{x^4 + x^2} = \frac{0x + (-1)}{x^2 + 1} + \frac{1}{x^2} + \frac{0}{x} = \frac{-1}{x^2+1} + \frac{1}{x^2}$$

Hence, using K for the generic constant to avoid confusing it with the C used above,

$$\begin{aligned}\int \frac{1}{x^4+x^2}\,dx &= \int \left(\frac{-1}{x^2+1}+\frac{1}{x^2}\right)dx = \int \frac{-1}{x^2+1}\,dx + \int \frac{1}{x^2}\,dx \\ &= -\int \frac{1}{x^2+1}\,dx + \int x^{-2}\,dx = -\arctan(x) + (-1)x^{-1} + K \\ &= -\arctan(x) - \frac{1}{x} + K\,. \quad \square\end{aligned}$$

Quiz #15. Wednesday, 13 July, 2011. *[10 minutes]*

1. Compute $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$. *[5]*

SOLUTION. This is an improper integral, so we need to take a limit:

$$\begin{aligned}\int_1^\infty \frac{1}{x^2}\,dx &= \lim_{t\to\infty}\int_1^t \frac{1}{x^2}\,dx = \lim_{t\to\infty}\int_1^t x^{-2}\,dx = \lim_{t\to\infty} -x^{-1}\Big|_1^t = \lim_{t\to\infty} -\frac{1}{x}\Big|_1^t \\ &= \lim_{t\to\infty}\left[-\frac{1}{t}-\left(-\frac{1}{1}\right)\right] = \lim_{t\to\infty}\left[-\frac{1}{t}+1\right] = -0+1 = 1\end{aligned}$$

since $\frac{1}{t} \to 0$ as $t \to \infty$. $\square$

Quiz #16. Monday, 18 July, 2011. *[12 minutes]*

Do *one* of questions 1 or 2.

1. Find the arc-length of the curve $y = \frac{2}{3}x^{3/2}$, where $0 \le x \le 3$. *[5]*

SOLUTION. First, $\displaystyle\frac{dy}{dx} = \frac{d}{dx}\frac{2}{3}x^{3/2} = \frac{2}{3}\cdot\frac{3}{2}x^{1/2} = x^{1/2} = \sqrt{x}$. Plugging this into the arc-length formula gives:

$$\begin{aligned}\text{arc-length} &= \int_0^3 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_0^3 \sqrt{1+\left(\sqrt{x}\right)^2}\,dx = \int_0^3 \sqrt{1+x}\,dx \\ &\qquad \text{Substitute } u = x+1, \text{ so } du = dx \text{ and } \begin{matrix} x & 0 & 3 \\ u & 1 & 4 \end{matrix}. \\ &= \int_1^4 \sqrt{u}\,du = \int_1^4 u^{1/2}\,du = \frac{2}{3}u^{3/2}\Big|_1^4 \\ &= \frac{2}{3}\left(4^{3/2}-1^{3/2}\right) = \frac{2}{3}(8-1) = \frac{14}{3} \quad \square\end{aligned}$$

2. Find the area of the surface of revolution obtained by rotating the curve $y = 1 - \frac{1}{2}x^2$, where $0 \le x \le \sqrt{3}$, about the y-axis. *[5]*

SOLUTION. First, $\frac{dy}{dx} = \frac{d}{dx}\left(1 - \frac{1}{2}x^2\right) = 0 - \frac{1}{2}2x = -x$. Second, since we are rotating the curve about the y-axis, $r = x - 0 = x$. Plugging these into the formula for the area of a surface of revolution gives:

$$\text{area} = \int_0^{\sqrt{3}} 2\pi x \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_0^{\sqrt{3}} 2\pi x \sqrt{1 + (-x)^2}\, dx = \pi \int_0^{\sqrt{3}} 2x\sqrt{1+x^2}\, dx$$

Substitute $u = 1 + x^2$, so $du = 2x\, dx$ and $\begin{array}{ccc} x & 0 & \sqrt{3} \\ u & 1 & 4 \end{array}$.

$$= \pi \int_1^4 \sqrt{u}\, du = \pi \int_1^4 u^{1/2}\, du = \pi\, \frac{2}{3}u^{3/2}\Big|_1^4$$

$$= \frac{2}{3}\pi\left(4^{3/2} - 1^{3/2}\right) = \frac{2}{3}\pi\,(8-1) = \frac{14}{3}\pi \quad \square$$

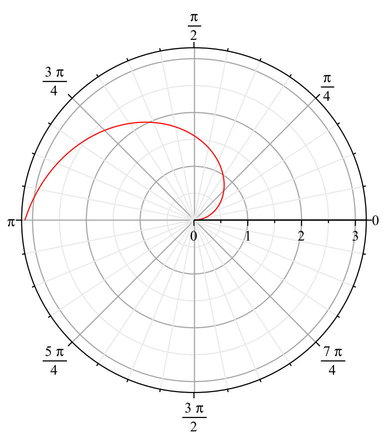
Quiz #17. Wednesday, 19 July, 2011. *[12 minutes]*

Do *one* of questions 1 or 2.

1. Sketch the curve $r = \theta$, $0 \le \theta \le \pi$, in polar coordinates and the area of the region between the curve and the origin. *[5]*

SOLUTION. Cheating a bit, we use `Maple` to graph the curve:

```
> plots[polarplot](theta,theta=0..Pi);
```



To find the area of the region between curve and the origin, we plug the curve into the area formula for polar regions:

$$A = \int_C \frac{1}{2}r^2\, d\theta = \int_0^\pi \frac{1}{2}\theta^2\, d\theta = \frac{1}{2}\cdot\frac{1}{3}\theta^3\Big|_0^\pi = \frac{1}{6}\pi^3 - \frac{1}{6}0^3 = \frac{1}{6}\pi^3 \quad \square$$

2. For which values of x does the series $\sum_{n=0}^{\infty} x^{n+2} = x^2 + x^3 + x^4 + \cdots$ converge? What is the sum when it does converge? *[5]*

SOLUTION. $\sum_{n=0}^{\infty} x^{n+2} = x^2 + x^3 + x^4 + \cdots$ is a geometric series with first term $a = x^2$ and common ratio $r = x$ between successive terms. It follows that the series converges if either $a = x^2 = 0$, *i.e.* $x = 0$, or $|r| = |x| < 0$, *i.e.* $-1 < x < 1$. Since $-1 < 0 < 1$, this means that the series converges exactly when $-1 < x < 1$.

When the series converges, it must converge to the value given by the formula for the sum of a geometric series, namely, $\dfrac{a}{1-r} = \dfrac{x^2}{1-x}$. □

Quiz #18. Monday, 25 July, 2011. *[12 minutes]*

1. Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2 + 2^n}$ converges or diverges. *[5]*

SOLUTION. We will use the (basic form of the) Comparison Test to show that the given series converges. Since increasing a denominator makes the fraction smaller, we have both

$$\frac{1}{n^2 + 2^n} < \frac{1}{n^2} \qquad \text{and} \qquad \frac{1}{n^2 + 2^n} < \frac{1}{2^n}$$

for all n. (Note that all the terms above will always be positive for $n \geq 1$.) Either comparison will do the trick:

Comparison i. The series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots$ converges by the Integral Test. As $f(x) = \dfrac{1}{x^2}$ is a positive, decreasing, and continuous function for $1 \leq x < \infty$ such that $\dfrac{1}{n^2} = f(n)$, the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges if and only if the improper integral $\displaystyle\int_1^{\infty} \frac{1}{x^2}\,dx$ converges. Since

$$\int_1^{\infty} \frac{1}{x^2}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^2}\,dx = \lim_{t\to\infty} -\frac{1}{x}\bigg|_1^t = \lim_{t\to\infty} \left[\left(-\frac{1}{t}\right) - \left(-\frac{1}{1}\right)\right] = (-0) - (-1) = 1$$

(note that $\frac{1}{t} \to 0$ as $t \to \infty$), it follows that the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges. In turn, it follows by the Comparison Test that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2 + 2^n}$ converges.

Comparison ii. The series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$ converges because it is a geometric series with common ratio $r = \dfrac{1}{2}$ and $\left|\dfrac{1}{2}\right| = \dfrac{1}{2} < 1$. It follows by the Comparison Test that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2 + 2^n}$ converges. □

Quiz #19. Wednesday, 27 July, 2011. *[12 minutes]*

Do *one* of questions 1 or 2.

1. Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^n}{(n+1)!}$ converges absolutely, converges conditionally, or diverges. *[5]*

SOLUTION. We will use the Ratio Test:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{\frac{(-2)^{n+1}}{((n+1)+1)!}}{\frac{(-2)^n}{(n+1)!}}\right| = \lim_{n\to\infty} \left|\frac{(-2)^{n+1}}{(n+2)!} \cdot \frac{(n+1)!}{(-2)^n}\right|$$
$$= \lim_{n\to\infty} \left|\frac{-2}{(n+2)}\right| = \lim_{n\to\infty} \frac{2}{n+2} = 0\,,$$

since $n+2 \to \infty$ as $n \to \infty$. $0 < 1$, so it follows from the Ratio Test that the given series converges absolutely. $\square$

2. Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n e^n}{n}$ converges absolutely, converges conditionally, or diverges. *[5]*

SOLUTION. We will try the Alternating Series Test, which requires checking that the series is indeed alternating, that the terms are decreasing in absolute value, and that the limit of the terms is 0.

First, we check if the series is indeed alternating. Since $n > 0$ and $e^n > 0$ for $n \geq 1$, the $(-1)^n$ forces the terms of the given series to alternate in sign. So far, so good.

Second, we check whether the terms decrease in absolute value. Is it the case that $\left|\frac{(-1)^{n+1}e^{n+1}}{n+1}\right| = \frac{e^{n+1}}{n+1} < \frac{e^n}{n} = \left|\frac{(-1)^n e^n}{n}\right|$ for all n (past some point)? Observe that

$$\frac{e^{n+1}}{n+1} < \frac{e^n}{n} \quad\iff\quad ne^{n+1} < (n+1)e^n \quad\iff\quad ne < n+1 \quad\iff\quad e < 1 + \frac{1}{n}\,,$$

which last is *not* true for *any* $n \geq 1$ because $e > 2 \geq 1 + \frac{1}{n}$. It follows that the Alternating Series Test cannot be used to conclude that the series converges. (Which, unfortunately, is not the same as being able to conclude that the series diverges.) Nevertheless, it pays to check the last condition required by the Alternating Series Test.

Third, we check whether the limit of the (absolute values of) the terms is 0:

$$\lim_{n\to\infty} \left|\frac{(-1)^n e^n}{n}\right| = \lim_{n\to\infty} \frac{e^n}{n} = \lim_{x\to\infty} \frac{e^x}{x} \quad \begin{matrix}\to\infty \\ \to\infty\end{matrix} \quad \text{as } x \to \infty\text{, so we use l'Hôpital's Rule.}$$
$$= \lim_{x\to\infty} \frac{\frac{d}{dx}e^x}{\frac{d}{dx}x} = \lim_{x\to\infty} \frac{e^x}{1} = \infty$$

Not only does this mean that the Alternating Series Test cannot be used to conclude that the series converges, but that the series diverges. Failing this part of the Alternating Series Test amounts to failing the Divergence Test, so it follows that the given series diverges. $\square$