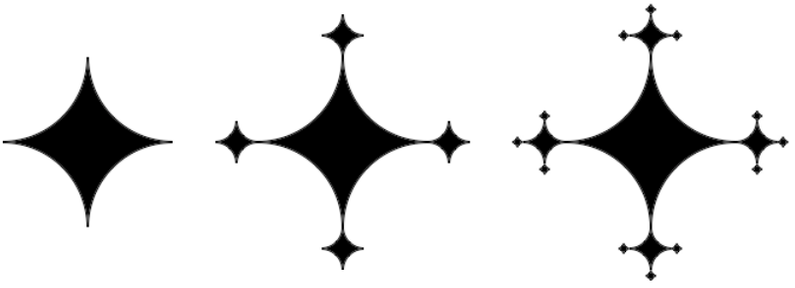
Mathematics 1100Y – Calculus I: Calculus of one variable
TRENT UNIVERSITY, SUMMER 2011

Assignment #11
Cross of Squircles
Due on Wednesday, 27 July, 2011.

Let's call the shape that you get by removing four mutually tangent quarter-circles with radius $\frac{s}{2}$ from a square with side length s a *squircle*[*]. (See the leftmost shape in the diagram below.)



A single squircle has four points where the quarter-circles that were removed met. Consider the following process:

At step $n = 0$ we have a single squircle for which $s = 2$.

At step $n = 1$, we attach four squircles for which $s = \frac{1}{4} \cdot 2 = \frac{1}{2}$ to the squircle in step 0, attaching one (at one of its points) to each point of the larger squircle. (See the middle shape in the diagram above.) The resulting shape has $3 \cdot 4 = 12$ points (where quarter-circles met) to which nothing is yet attached. Let's call these the *free* points of the shape.

At step $n = 2$, we attach a squircle for which $s = \frac{1}{4} \cdot \frac{1}{2} = \left(\frac{1}{4}\right)^2 \cdot 2 = \frac{1}{8}$ to each of the free points in the shape in step 1. (See the rightmost shape in the diagram above.) The resulting shape has $3 \cdot 12 = 3 \cdot (3 \cdot 4) = 3^2 \cdot 4 = 36$ free points.

At step $n = 3$, we attach a squircle for which $s = \frac{1}{4} \cdot \frac{1}{8} = \left(\frac{1}{4}\right)^3 \cdot 2 = \frac{1}{32}$ to each of these the free points in the shape in step 2. (Draw your own picture!) The resulting shape has $3 \cdot 36 = 3 \cdot \left(3^2 \cdot 4\right) = 3^3 \cdot 4 = 108$ free points.

Repeat for each integer $n > 3$ ...

1. Find formulas for the values of s for the squircles added at step n and for the number of free points of the shape obtained in step n. *[1]*
2. Find a formula for the length of the perimeter (*i.e.* border) of the shape obtained in step n. *[1.5]*
3. Find a formula for the area of the shape obtained in step n. *[1.5]*
4. Compute the length of the perimeter of the shape obtained after infinitely many steps of the process. *[3]*
5. Compute the area of the shape obtained after infinitely many steps of the process. *[3]*

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[*] No doubt this shape already has a name, but I don't know it ...