Mathematics 1100Y – Calculus I: Calculus of one variable

Trent University, Summer 2010

Solutions to Test 2

Instructions

- *Show all your work.* Legibly, please!
- *If you have a question, ask it!*
- Use the back sides of the test sheets for rough work or extra space.
- You may use a calculator and an aid sheet.

1. Compute any *four* (4) of the integrals in parts **a**-**f**. *[16 = 4 × 4 each]*

a. $\displaystyle\int \frac{1}{4-x^2}\,dx$ **b.** $\displaystyle\int \tan(x)\,dx$ **c.** $\displaystyle\int_0^1 \frac{1}{\sqrt{x}}\,dx$

d. $\displaystyle\int \frac{x^3+x+1}{x^2+1}\,dx$ **e.** $\displaystyle\int_{-\pi/4}^{\pi/4} \sec^2(x)\,dx$ **f.** $\displaystyle\int x\ln(x)\,dx$

Solution i to **a**. *(Partial fractions.)* Since $\dfrac{1}{4-x^2} = \dfrac{1}{(2-x)(2+x)} = \dfrac{A}{2-x} + \dfrac{B}{2+x}$, we have $1 = A(2+x) + B(2-x) = (A-B)x + (2A+2B)$. This boils down to solving the linear equations $A - B = 0$, *i.e.* $A = B$, and $2A + 2B = 1$, *i.e.* $A + B = \frac{1}{2}$. Plugging the first into the second gives $2A = \frac{1}{2}$; it follows that $B = A = \frac{1}{4}$. Now

$$\int \frac{1}{4-x^2}\,dx = \frac{1}{4}\int \frac{1}{2-x}\,dx + \frac{1}{4}\int \frac{1}{2+x}\,dx$$

Substitute $u = 2 - x$, so $du = -dx$ and $dx = (-1)\,du$, in the first integral, and $w = 2 + x$, so $dw = dx$, in the second.

$$= \frac{1}{4}\int \frac{-1}{u}\,du + \frac{1}{4}\int \frac{1}{w}\,dw = -\frac{1}{4}\ln(u) + \frac{1}{4}\ln(w) + C$$

$$= -\frac{1}{4}\ln(2-x) + \frac{1}{4}\ln(2+x) + C\,. \qquad \blacksquare$$

Solution ii to **a**. *(Trig substitution.)* We'll use the trigonometric substitution $x = 2\sin(\theta)$, so $dx = 2\cos(\theta)\,d\theta$. Note that it follows that $\sin(\theta) = \frac{x}{2}$ and $\cos(\theta) = \sqrt{1-\frac{x^2}{4}}$. Now

$$\int \frac{1}{4-x^2}\,dx = \int \frac{2\cos(\theta)}{4-4\sin^2(\theta)}\,d\theta = \int \frac{2\cos(\theta)}{4\cos^2(\theta)}\,d\theta = \int \frac{1}{2\cos(\theta)}\,d\theta = \frac{1}{2}\int \sec(\theta)\,d\theta$$

$$= \frac{1}{2}\ln\left(\sec(\theta)+\tan(\theta)\right) + C = \frac{1}{2}\ln\left(\frac{1}{\cos(\theta)} + \frac{\sin(\theta)}{\cos(\theta)}\right) + C$$

$$= \frac{1}{2}\ln\left(\frac{1}{\sqrt{1-\frac{x^2}{4}}} + \frac{\frac{x}{2}}{\sqrt{1-\frac{x^2}{4}}}\right) + C = \frac{1}{2}\ln\left(\frac{1+\frac{x}{2}}{\sqrt{1-\frac{x^2}{4}}}\right) + C\,. \qquad \blacksquare$$

Exercise: Show that solutions I and II to **a** actually give the same answer.

Solution to **b**. We'll use the definition of $\tan(x)$:

$$\begin{aligned}\int \tan(x)\,dx &= \int \frac{\sin(x)}{\cos(x)}\,dx \qquad \text{Substitute } u=\cos(x)\text{, so } du=-\sin(x)\,dx \text{ and } (-1)\,du=\sin(x)\,dx. \\ &= \int \frac{-1}{u}\,du = -\ln(u)+C = -\ln(\cos(x))+C = \ln\left(\frac{1}{\cos(x)}\right)+C \\ &= \ln(\sec(x))+C \qquad \blacksquare\end{aligned}$$

Solution to **c**. We'll use the Power Rule:

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx = \int_0^1 x^{-1/2}\,dx = \left.\frac{x^{1/2}}{1/2}\right|_0^1 = 2\sqrt{x}\Big|_0^1 = 2\sqrt{1}-2\sqrt{0} = 2 \qquad \blacksquare$$

Exercise: Explain why the method used in **c** is not quite correct, even though it gives the right answer.

Solution to **d**. Note that $\dfrac{x^3+x+1}{x^2+1}$ is a rational function in which the degree of the denominator is less that the degree of the numerator. Since $x^3+x+1 = x\left(x^2+1\right)+1$ (you can do long division to get this, or just use the "eyeball theorem"), it follows that

$$\frac{x^3+x+1}{x^2+1} = \frac{x\left(x^2+1\right)+1}{x^2+1} = x + \frac{1}{x^2+1}\,.$$

Hence

$$\begin{aligned}\int \frac{x^3+x+1}{x^2+1}\,dx &= \int \left(x+\frac{1}{x^2+1}\right)dx = \int x\,dx + \int \frac{1}{x^2+1}\,dx \\ &= \frac{1}{2}x^2 + \arctan(x) + C\,.\end{aligned}$$

Those who haven't yet memorized that the antiderivative of $\dfrac{1}{x^2+1}$ is $\arctan(x)$ can get it with the trig substitution $x=\tan(\theta)$. $\blacksquare$

Solution to **e**.

$$\int_{-\pi/4}^{\pi/4} \sec^2(x)\,dx = \tan(x)\big|_{-\pi/4}^{\pi/4} = \tan(\pi/4) - \tan(-\pi/4) = 1-(-1) = 2$$

since $\sin(\pi/4)=\cos(\pi/4)=\cos(-\pi/4)=\frac{1}{\sqrt{2}}$ and $\sin(-\pi/4) = -\frac{1}{\sqrt{2}}$. $\blacksquare$

SOLUTION TO **f**. We'll do this one using integration by parts; let $u = \ln(x)$ and $v' = x$, so $u' = \frac{1}{x}$ and $v = \frac{1}{2}x^2$.

$$\int x\ln(x)\,dx = \ln(x)\cdot\frac{1}{2}x^2 - \int \frac{1}{x}\cdot\frac{1}{2}x^2\,dx = \frac{1}{2}x^2\ln(x) - \frac{1}{2}\int x\,dx$$
$$= \frac{1}{2}x^2\ln(x) - \frac{1}{2}\cdot\frac{1}{2}x^2 + C = \frac{1}{2}x^2\ln(x) - \frac{1}{4}x^2 + C \qquad \blacksquare$$

2. Do any *two* (2) of parts **a-e**. *[12 = 2 × 6 each]*

a. Compute $\int_0^2 (x+1)\,dx$ using the Right-hand Rule.

SOLUTION. We plug into the formula and chug away:

$$\int_0^2 (x+1)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{2-0}{n}\cdot\left[\left(0+i\frac{2-0}{n}\right)+1\right] = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{2}{n}\cdot\left(i\frac{2}{n}+1\right)$$
$$= \lim_{n\to\infty}\frac{2}{n}\sum_{i=1}^{n}\left(i\frac{2}{n}+1\right) = \lim_{n\to\infty}\frac{2}{n}\left(\left[\sum_{i=1}^{n} i\frac{2}{n}\right]+\left[\sum_{i=1}^{n}1\right]\right)$$
$$= \lim_{n\to\infty}\frac{2}{n}\left(\left[\frac{2}{n}\sum_{i=1}^{n} i\right]+n\right) = \lim_{n\to\infty}\frac{2}{n}\left(\frac{2}{n}\cdot\frac{n(n+1)}{2}+n\right)$$
$$= \lim_{n\to\infty}\frac{2}{n}\left((n+1)+n\right) = \lim_{n\to\infty}\frac{2}{n}(2n+1)$$
$$= \lim_{n\to\infty}\left(4+\frac{2}{n}\right) = 4+0 = 4\,,$$

since $\frac{2}{n}\to 0$ as $n\to\infty$. $\blacksquare$

b. Find the area of the region bounded by $y = 2+x$ and $y = x^2$ for $-1 \le x \le 1$.

SOLUTION. A little experimentation with values at different points or a very quick sketch suffices to show that the two curves touch at $x = -1$ and that $2+x > x^2$ until somewhere to the right of $x = 1$. Thus the area between the two curves for $-1 \le x \le 1$ is:

$$\int_{-1}^{1}\left(2+x-x^2\right)\,dx = \left(2x+\frac{x^2}{2}-\frac{x^3}{3}\right)\Bigg|_{-1}^{1}$$
$$= \left(2\cdot 1+\frac{1^2}{2}-\frac{1^3}{3}\right) - \left(2\cdot(-1)+\frac{(-1)^2}{2}-\frac{(-1)^3}{3}\right)$$
$$= \left(2+\frac{1}{2}-\frac{1}{3}\right) - \left(-2+\frac{1}{2}+\frac{1}{3}\right)$$
$$= \frac{13}{6} - \frac{-5}{6} = \frac{18}{6} = 3 \qquad \blacksquare$$

c. Without actually computing $\int_0^{10/\pi} \arctan(x)\, dx$, find as small an upper bound as you can on the value of this integral.

SOLUTION. Note that $\arctan(x) < \frac{\pi}{2}$ for all x. (Just look at the graph!) It follows that

$$\int_0^{10/\pi} \arctan(x)\, dx < \int_0^{10/\pi} \frac{\pi}{2}\, dx = \frac{\pi}{2} x \Big|_0^{10/\pi} = \frac{\pi}{2} \cdot \frac{10}{\pi} - \frac{\pi}{2} \cdot 0 = 5\,.$$

Good enough for me! ■

d. Compute the arc-length of the curve $y = \ln(\cos(x))$, $0 \le x \le \pi/6$.

SOLUTION. We plug into the arc-length formula and chug away. Note that

$$\frac{dy}{dx} = \frac{1}{\cos x} \cdot \frac{d}{dx} \cos(x) = \frac{1}{\cos x} \cdot (-\sin(x)) = -\frac{\sin(x)}{\cos x} = -\tan(x)\,,$$

and that $\cos(\pi/6) = \frac{1}{2}$ and $\sin(\pi/6) = \frac{\sqrt{3}}{2}$, so $\sec(\pi/6) = 2$ and $\tan(\pi/6) = \sqrt{3}$. Then

$$\begin{aligned}
\text{arc length} &= \int_0^{\pi/6} \sqrt{1 + (-\tan(x))^2}\, dx = \int_0^{\pi/6} \sqrt{1 + \tan^2(x)}\, dx \\
&= \int_0^{\pi/6} \sqrt{\sec^2(x)}\, dx = \int_0^{\pi/6} \sec(x)\, dx = \ln(\sec(x) + \tan(x))\big|_0^{\pi/6} \\
&= \ln(\sec(\pi/6) + \tan(\pi/6)) - \ln(\sec(0) + \tan(0)) \\
&= \ln\left(2 + \sqrt{3}\right) - \ln(1 + 0) = \ln\left(2 + \sqrt{3}\right)
\end{aligned}$$

since $\ln(1) = 0$. ■

NOTE: If you want to know where $y = \ln(\cos(x))$ came from, look at the solution to **1b**.

e. Give a example of a function $f(x)$ such that $f(x) = 1 + \int_0^x f(t)\, dt$ for all x.

SOLUTION. Suppose $f(x)$ satisfied the given equation. Then, by the Fundamental Theorem of Calculus, we would have that

$$f'(x) = \frac{d}{dx}\left(1 + \int_0^x f(t)\, dt\right) = 0 + f(x) = f(x)\,.$$

One well-known function satisfying this condition is $f(x) = 0$, but it fails to satisfy the original equation since

$$0 \ne 1 = 1 + 0 = 1 + \int_0^0 0\, dt\,.$$

The other well-known function satisfying $f'(x) = f(x)$ is $f(x) = e^x$. Since

$$e^0 = 1 = 1 + 0 = 1 + \int_0^0 e^t\, dt\,,$$

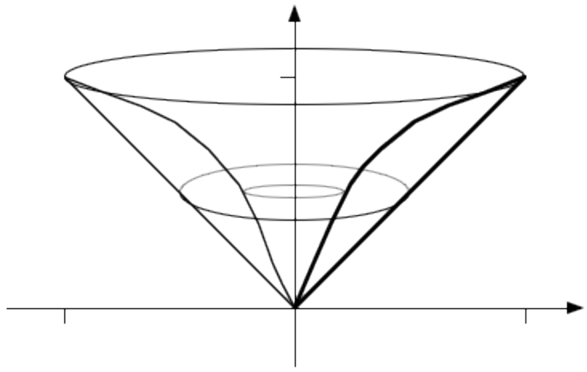
it has chance. We verify that it does satisfy the original equation:

$$1 + \int_0^x e^t\, dt = 1 + e^t\big|_0^x = 1 + \left(e^x - e^0\right) = 1 + e^x - 1 = e^x \qquad ■$$

3. Do *one* (1) of parts **a** or **b**. *[12]*

a. Sketch the solid obtained by rotating the region bounded by $y = \sqrt{x}$ and $y = x$, where $0 \le x \le 1$, about the y-axis, and find its volume.

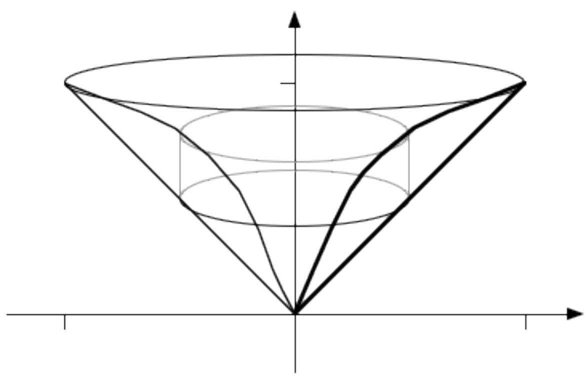
SOLUTION I TO **a**. *(Disks/Washers)* Note that we have $x \le \sqrt{x}$ for $0 \le x \le 1$. Here's a sketch of the solid, with a typical "washer" cross-section in the picture.



Considering the sketch, it is easy to see that the washer at height y has an out radius of $R = x = y$ and an inner radius of $r = x = y^2$ (since $y = \sqrt{x}$), and hence area $\pi\left(R^2 - r^2\right) = \pi\left(y^2 - \left(y^2\right)^2\right) = \pi\left(y^2 - y^4\right)$. Since we rotated the region about the y-axis, the washers are stacked vertically, so we must integrate over y to get the volume of the solid. Note that $0 \le y \le 1$ over the given region.

$$\text{Volume} = \int_0^1 \pi\left(R^2 - r^2\right)\,dy = \pi \int_0^1 \left(y^2 - y^4\right)\,dy = \pi\left(\frac{y^3}{3} - \frac{y^5}{5}\right)\Big|_0^1$$
$$= \pi\left(\frac{1^3}{3} - \frac{1^5}{5}\right) - \pi\left(\frac{0^3}{3} - \frac{0^5}{5}\right) = \pi\left(\frac{1}{3} - \frac{1}{5}\right) - \pi(0 - 0) = \frac{2}{15}\pi \qquad \blacksquare$$

SOLUTION II TO **a**. *(Cylindrical shells)* Note that we have $x \le \sqrt{x}$ for $0 \le x \le 1$. Here's a sketch of the solid, with a typical cylindrical shell in the picture.

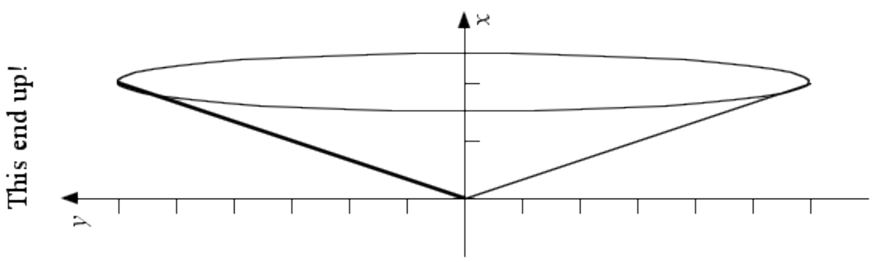


Considering the sketch, it is easy to see that the cylinder centred on the y-axis with radius $r = x$ has height $h = \sqrt{x} - x$, and hence area $2\pi rh = 2\pi x\left(\sqrt{x} - x\right)$. Since we rotated the region about the y-axis, the cylinders are vertical and so nested horizontally, so we must integrate over x to get the volume of the solid.

$$\begin{aligned}
\text{Volume} &= \int_0^1 2\pi rh\,dx = 2\pi \int_0^1 x\left(\sqrt{x} - x\right)\,dx = 2\pi \int_0^1 \left(x^{3/2} - x^2\right)\,dx \\
&= 2\pi\left(\frac{x^{5/2}}{5/2} - \frac{x^3}{3}\right)\Bigg|_0^1 = 2\pi\left(\frac{1^{5/2}}{5/2} - \frac{1^3}{3}\right) - 2\pi\left(\frac{0^{5/2}}{5/2} - \frac{0^3}{3}\right) \\
&= 2\pi\left(\frac{2}{5} - \frac{1}{3}\right) - 2\pi(0-0) = 2\pi\frac{1}{15} = \frac{2}{15}\pi \qquad \blacksquare
\end{aligned}$$

b. Sketch the cone obtained by rotating the line $y = 3x$, where $0 \le x \le 2$, about the x-axis, and find its surface area.

SOLUTION. Here's a sketch of the cone:



The cross-section of the cone at x has radius $r = y = 3x$ and $\frac{dy}{dx} = 3$. Hence

$$\begin{aligned}
\text{Surface Area} &= \int_0^2 2\pi r\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = 2\pi\int_0^2 3x\sqrt{1+3^2}\,dx \\
&= 6\sqrt{10}\,\pi\int_0^2 x\,dx = 6\sqrt{10}\,\pi\,\frac{x^2}{2}\Bigg|_0^2 = 6\sqrt{10}\,\pi\left(\frac{2^2}{2} - \frac{0^2}{2}\right) \\
&= 6\sqrt{10}\,\pi(2-0) = 12\sqrt{10}\,\pi\,. \qquad \blacksquare
\end{aligned}$$

[Total = 40]