Mathematics 1110H (Section A) – Calculus I: Limits, Derivatives, and Integrals
Trent University, Fall 2024

Assignment #4 (typos corrected)
Sums
*Due on Friday, 8 November.**

If you haven't already seen them, please look up SageMath's `sum` and `limit` commands before tackling this assignment. By way of notation, if $f(k)$ is some function of the integer variable k, then the expression $\sum_{k=a}^{b} f(k)$ is shorthand for the sum $f(a) + f(a+1) + f(a+2) + \cdots + f(b)$. For example, if $f(k) = 1$ for all k, then

$$\sum_{k=1}^{n} 1 = \underbrace{! + 1 + \cdots + 1}_{n \text{ copies of 1 added up}} .$$

This sum, of course, adds up to n; in the professional lingo, its *summation formula* is n.

1. Use SageMath to find a summation formula in terms of n for each of the following sums:

a. $\sum_{k=1}^{n} k$ *[0.5]* **b.** $\sum_{k=1}^{n} k^2$ *[0.5]* **c.** $\sum_{k=1}^{n} k^3$ *[0.5]* **d.** $\sum_{k=1}^{n} k^4$ *[0.5]*

2. Give an argument that verifies that the summation formula SageMath gave you for $\sum_{k=1}^{n} k$ is true for all $n \geq 1$. *[2]*

Hint: There is a cheap algebraic trick available here. Carl Friedrich Gauss (1777-1855), usually counted as one of the greatest mathematicians ever, is supposed to have used it to compute the sum $1 + 2 + \cdots + 100$ as a child.

One can also try to add up sums of infinitely many numbers. Technically, these are limits of finite sums, *i.e.* $\sum_{k=1}^{\infty} f(k) = \lim_{n\to\infty} \sum_{k=1}^{n} f(k)$. Be advised that in many cases infinite sums do not add up to a real number. For example, $\sum_{k=1}^{\infty} 1 = \lim_{n\to\infty} \sum_{k=1}^{n} 1 = \lim_{n\to\infty} n = \infty$. For another example, note that $\sum_{k=1}^{n} (-1)^{k+1} = 1 - 1 + 1 - \cdots + (-1)^{n+1} = \begin{cases} 1 & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$.

* Please submit your solutions, preferably as a single pdf, via Blackboard's Assignments module. If that fails, please submit them to the instructor on paper or via email to `sbilaniuk@trentu.ca` as soon as you can.

Since the finite sums alternate between two different numbers, their limit as $n \to \infty$ does not exist, which means the corresponding infinite sum does not add up to any real number.

3. Consider the infinite sum $\displaystyle\sum_{k=0}^{\infty} \frac{1}{2^k} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$.

- **a.** Explain why this infinite sum adds up to 2. *[0.5]*
- **b.** Use SageMath to find a summation formula in terms of n for the finite sum $\displaystyle\sum_{k=0}^{n} \frac{1}{2^k}$. *[0.5]*
- **c.** Use SageMath to compute the infinite sum by taking the limit as $n \to \infty$ of the formula you obtained in part **b**. *[0.5]*
- **c.** Use SageMath to compute the infinite sum directly. *[0.5]*

4. Consider the infinite sum $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2+k} = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \cdots$.

- **a.** Explain why this infinite sum adds up to 1. *[1]*
- **b.** Use your algebra or SageMath skills to find a summation formula in terms of n for the finite sum $\displaystyle\sum_{k=0}^{n} \frac{1}{k^2+k}$. *[0.5]*
- **c.** Whether by hand or SageMath, compute the infinite sum by taking the limit as $n \to \infty$ of the formula you obtained in part **b**. *[0.5]*
- **c.** Use SageMath to compute the infinite sum directly. *[0.5]*

5. What does the infinite sum $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k}$ add up to? Explain why. *[1.5]*

Hint: Don't forget that you can look things up ...