Mathematics 1110H (Section A) – Calculus I: Limits, Derivatives, and Integrals
Trent University, Fall 2024

Solutions to Assignment #4
Sums

If you haven't already seen them, please look up SageMath's `sum` and `limit` commands before tackling this assignment. By way of notation, if $f(k)$ is some function of the integer variable k, then the expression $\sum_{k=a}^{b} f(k)$ is shorthand for the sum $f(a) + f(a+1) + f(a+2) + \cdots + f(b)$. For example, if $f(k) = 1$ for all k, then

$$\sum_{k=1}^{n} 1 = \underbrace{! + 1 + \cdots + 1}_{n \text{ copies of 1 added up}} .$$

This sum, of course, adds up to n; in the professional lingo, its *summation formula* is n.

1. Use SageMath to find a summation formula in terms of n for each of the following sums:

a. $\sum_{k=1}^{n} k$ *[0.5]* **b.** $\sum_{k=1}^{n} k^2$ *[0.5]* **c.** $\sum_{k=1}^{n} k^3$ *[0.5]* **d.** $\sum_{k=1}^{n} k^4$ *[0.5]*

Solutions. **a.** SageMath gives us:

```
[1]: var('n')
     var('k')
     sum( k, k, 1, n )
```

```
[1]: 1/2*n^2 + 1/2*n
```

That is, $\sum_{k=1}^{n} k = \frac{1}{2} \cdot n^2 + \frac{1}{2} \cdot n$. This formula is usually given as $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$. □

b. SageMath gives us:

```
[2]: sum( k^2, k, 1, n )
```

```
[2]: 1/3*n^3 + 1/2*n^2 + 1/6*n
```

That is, $\sum_{k=1}^{n} k^2 = \frac{1}{3} \cdot n^3 + \frac{1}{2} \cdot n^2 + \frac{1}{6} \cdot n$, usually given as $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$. □

c. SageMath gives us:

```
[3]: sum( k^3, k, 1, n )
```

```
[3]: 1/4*n^4 + 1/2*n^3 + 1/4*n^2
```

That is, $\sum_{k=1}^{n} k^3 = \frac{1}{4} \cdot n^4 + \frac{1}{2} \cdot n^3 + \frac{1}{4} \cdot n^2$, usually given as $\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$. $\square$

d. SageMath gives us:

```
[4]: sum( k^4, k, 1, n )

[4]: 1/5*n^5 + 1/2*n^4 + 1/3*n^3 - 1/30*n
```

That is, $\sum_{k=1}^{n} k^4 = \frac{1}{5} \cdot n^5 + \frac{1}{2} \cdot n^4 + \frac{1}{3} \cdot n^3 - \frac{1}{30} \cdot n$. If you want to see how this is usually given, look it up ... $\blacksquare$

2. Give an argument that verifies that the summation formula SageMath gave you for $\sum_{k=1}^{n} k$ is true for all $n \geq 1$. *[2]*

Hint: There is a cheap algebraic trick available here. Carl Friedrich Gauss (1777-1855), usually counted as one of the greatest mathematicians ever, is supposed to have used it to compute the sum $1 + 2 + \cdots + 100$ as a child.

SOLUTION. Since

$$\begin{aligned} 2\sum_{k=1}^{n} k &= \quad 1 + 2 + \cdots + (n-1) + n \\ &\quad + n + (n-1) + \cdots + 2 + 1 \\ &= (n+1) + (n+1) + \cdots + (n+1) + (n+1) \\ &= n(n+1), \end{aligned}$$

it follows that $\sum_{k=1}^{n} k = \frac{n(n+1)}{2} = \frac{1}{2} \cdot n^2 + \frac{1}{2} \cdot n$. $\blacksquare$

One can also try to add up sums of infinitely many numbers. Technically, these are limits of finite sums, *i.e.* $\sum_{k=1}^{\infty} f(k) = \lim_{n\to\infty} \sum_{k=1}^{n} f(k)$. Be advised that in many cases infinite sums do not add up to a real number. For example, $\sum_{k=1}^{\infty} 1 = \lim_{n\to\infty} \sum_{k=1}^{n} 1 = \lim_{n\to\infty} n = \infty$. For another example, note that $\sum_{k=1}^{n} (-1)^{k+1} = 1 - 1 + 1 - \cdots + (-1)^{n+1} = \begin{cases} 1 & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$. Since the finite sums alternate between two different numbers, their limit as $n \to \infty$ does not exist, which means the corresponding infinite sum does not add up to any real number.

3. Consider the infinite sum $\displaystyle\sum_{k=0}^{\infty} \frac{1}{2^k} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$.

- **a.** Explain why this infinite sum adds up to 2. *[0.5]*
- **b.** Use SageMath to find a summation formula in terms of n for the finite sum $\displaystyle\sum_{k=0}^{n} \frac{1}{2^k}$. *[0.5]*
- **c.** Use SageMath to compute the infinite sum by taking the limit as $n \to \infty$ of the formula you obtained in part **b**. *[0.5]*
- **c.** Use SageMath to compute the infinite sum directly. *[0.5]*

Solutions. **a.** Each number added in this sum is half the remaining distance to 2. After infinitely many steps in this process we must be as close as possible to (that is, be right at) 2. □

b. SageMath saith:

```
[5]: sum( 1/2^k, k, 0, n )

[5]: (2^(n + 1) - 1)/2^n
```

That is, $\displaystyle\sum_{k=0}^{n} \frac{1}{2^k} = \frac{2^{n+1} - 1}{2^n}$. This is usually easier to use if rewritten as $\displaystyle\sum_{k=0}^{n} \frac{1}{2^k} = 2 - \frac{1}{2^n}$. □

c. Here we go:

```
[6]: limit( (2^(n + 1) - 1)/2^n, n=oo )

[6]: 2
```

Thus $\displaystyle\sum_{k=0}^{\infty} \frac{1}{2^k} = 2$. □

d. Here we go:

```
[7]: sum( 1/2^k, k, 0, oo )

[7]: 2
```

Thus, once again, $\displaystyle\sum_{k=0}^{\infty} \frac{1}{2^k} = 2$. ■

4. Consider the infinite sum $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2+k} = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \cdots$.

a. Explain why this infinite sum adds up to 1. *[1]*

b. Use your algebra or SageMath skills to find a summation formula in terms of n for the finite sum $\displaystyle\sum_{k=0}^{n} \frac{1}{k^2+k}$. *[0.5]*

c. Whether by hand or SageMath, compute the infinite sum by taking the limit as $n \to \infty$ of the formula you obtained in part **b**. *[0.5]*

c. Use SageMath to compute the infinite sum directly. *[0.5]*

SOLUTIONS. **a.** Algebraic trickery! Observe: $\dfrac{1}{k} - \dfrac{1}{k+1} = \dfrac{1(k+1) - 1k}{k(k+1)} = \dfrac{1}{k^2+k}$. It follows that

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{1}{k^2+k} = \sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+1}\right) &= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots \\ &= 1 + \left(-\frac{1}{2} + \frac{1}{2}\right) + \left(-\frac{1}{3} + \frac{1}{3}\right) + \left(-\frac{1}{4} + \frac{1}{4}\right) + \cdots \\ &= 1 + 0 + 0 + 0 + \cdots = 1. \quad \square\end{aligned}$$

b. *By hand.* Using the same trickery used for part **a** above, we have:

$$\begin{aligned}\sum_{k=1}^{n} \frac{1}{k^2+k} = \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1}\right) &= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right) \\ &= 1 + \left(-\frac{1}{2} + \frac{1}{2}\right) + \left(-\frac{1}{3} + \frac{1}{3}\right) + \cdots + \left(-\frac{1}{n} + \frac{1}{n}\right) - \frac{1}{n+1} \\ &= 1 + 0 + 0 + \cdots + 0 - \frac{1}{n+1} = 1 - \frac{1}{n+1} \quad \square\end{aligned}$$

b. *Using SageMath.*

```
[8]: sum( 1/(k^2 + k), k, 1, n )
```

```
[8]: n/(n + 1)
```

That is, $\displaystyle\sum_{k=1}^{n} \frac{1}{k^2+k} = \frac{n}{n+1}$. It's not hard to check that this is equal to the formula obtained above hand. $\square$

c. *By hand.* Here we go, using the summation formula obtained by hand for part **b**:

$$\sum_{k=1}^{\infty} \frac{1}{k^2+k} = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k^2+k} = \lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right) = 1 - 0 = 1 \quad \square$$

c. *Using SageMath.* ... and the summation formula that SageMath gave us for part **b**, because copy-paste makes that easy:

```
[9]: limit( n/(n + 1), n=oo )

[9]: 1
```

That is, $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2+k} = 1$. □

d. Here we go:

```
[10]: sum( 1/(k^2 - k), k, 1, oo )

[10]: 1
```

That is, $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2+k} = 1$. ■

5. What does the infinite sum $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k}$ add up to? Explain why. *[1.5]*

Hint: Don't forget that you can look things up ...

SOLUTIONS. Being lazy, we'll try throwing this sum into SageMath first:

```
[11]: sum( 1/k, k, 1, oo )

---------------------------------------------------------------------------
    RuntimeError                              Traceback (most recent call
 last)

    /opt/conda/envs/sage/lib/python3.7/site-packages/sage/interfaces/
 maxima_lib.py in sr_sum(self, *args)
    875         try:
--> 876             return
```

The error messages go on for a fair while longer and aren't very useful to us until the last line,

```
ValueError: Sum is divergent.
```

which tells us that the sum does not add up to a real number, though it doesn't tell us why.

Here is one reason why the given sum does not add up to a real number. Observe that :

$$
\begin{aligned}
1 &\geq \frac{1}{2} \\
\frac{1}{2} &\geq \frac{1}{2} \\
\frac{1}{3} + \frac{1}{4} &\geq \frac{1}{4} + \frac{1}{4} = 2 \cdot \frac{1}{4} = \frac{1}{2} \\
\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} &\geq \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = 4 \cdot \frac{1}{8} = \frac{1}{2} \\
\frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16} &\geq \frac{1}{16} + \cdots + \frac{1}{16} = 8 \cdot \frac{1}{16} = \frac{1}{2} \\
&\vdots \\
\frac{1}{2^m + 1} + \frac{1}{2^m + 2} + \cdots + \frac{1}{2^{m+1}} &\geq \frac{1}{2^{m+1}} + \cdots + \frac{1}{2^{m+1}} = 2^m \cdot \frac{1}{2^{m+1}} = \frac{1}{2} \\
&\vdots
\end{aligned}
$$

By grouping the numbers in the given infinite sum as indicated above, we get that

$$\sum_{k=1}^{\infty} \frac{1}{k} \geq \sum_{m=1}^{\infty} \frac{1}{2} = \lim_{n \to \infty} \sum_{m=1}^{n} \frac{1}{2} = \lim_{n \to \infty} \frac{n \to \infty}{2 \to 2} = \infty.$$

Since the given infinite sum is at least as big as another that "adds up to infinity", which is one way of not adding up to a real number or being *divergent*, it follows that the given infinite sum also "adds up to infinity", and hence not to any real number. ■