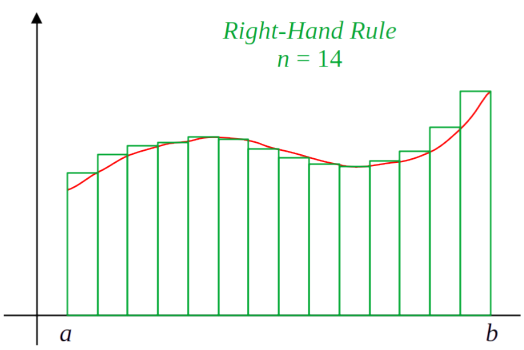
Right-Hand Rule Riemann Sums

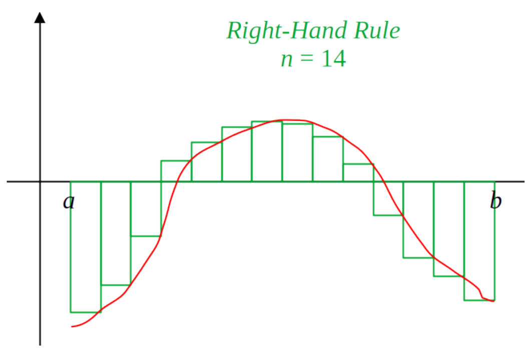
Integration is the more-or-less opposite operation to differentiation, though this is not exactly obvious from what they intutively tell us. Recall that differentiation gives us the slope of the tangent line at some point on the graph of a function. Intuitively, the definite integral $\int_a^b f(x)\,dx$ represents the area between $y = f(x)$ and the x-axis (*i.e.* $y = 0$, for $a \le x \le b$, weighted so that area above the x-axis is positive but the area below the x-axis is negative. The fact that finding these weighted areas and finding the slopes of tangent lines are pretty much inverse operations is essentially what the Fundamental Theorem of Calculus is all about. Before one can really prove that, or even state it precisely, one needs to have some definition of what the weighted area between the graph of a function and the x-axis

There are a number of ways to define the definite integral precisely. The first rigorous definition was due to Bernhard Riemann (1826-1866), who presented it in a talk in 1854, though it wasn't published until 1868. His basic idea was to approximate the area between $y = f(x)$ and the x-axis by using rectangles. As one makes the rectangles narrower and increases their number, their collective areas give better approximations to the area between the graph and the x-axis. Taking a suitable limit as one increases the number of rectangles and shrinks their maximum width lets one use this idea to define the definite integral. Unfortunately, to get the most out of this idea – being able to handle a lot of functions, as well as making it less inconvenient to prove the things one needs to about and using this definition – one *has to* allow variation in the widths of the rectangles and in how their heights are determined, resulting in a lot of additional complexity in the definition. (See the handout *A Precise Definition of the Definite Integral* for the simplest version of the general definition of the Riemann integral known to your instructor.) However, if one is willing to accept the cost of limiting which functions can be handled and making it more difficult to use the definition to prove general facts, including the Fundamental Theorem of Calculus, one simplify the general definition in various ways that make it easier to set up and understand, and perhaps to actually use to compute definite integrals. One such simplification is the so-called Right-Hand Rule, described below.

The Right-Hand Rule discards a lot of the complications of general Riemann sums of areas of rectangles by making all the rectangles have equal width and by determining the height of each rectangle by evaluating the function at the right-hand endpoint of its base. Thus if one approximated the weighted area between $y = f(x)$ and the x-axis for $a \le x \le b$, with, say 14 rectangles, it might look something like



or



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If you divide the interval $[a, b]$ into n equal pieces to serve as the bases of the rectangles, each piece will have width $\dfrac{b-a}{n}$, and these pieces will be subintervals of $[a, b]$, namely:

$$\left[a+0\frac{b-a}{n}, a+1\frac{b-a}{n}\right], \left[a+\frac{b-a}{n}, a+2\frac{b-a}{n}\right], \left[a+2\frac{b-a}{n}, a+3\frac{b-a}{n}\right],$$
$$\ldots, \left[a+(n-2)\frac{b-a}{n}, a+(n-1)\frac{b-a}{n}\right], \left[a+(n-1)\frac{b-a}{n}, a+n\frac{b-a}{n}\right]$$

Note that $a+0\dfrac{b-a}{n}=a$ and $a+n\dfrac{b-a}{n}=b$, and that the right-hand endpoint of the ith interval is $a+i\dfrac{b-a}{n}$. If we evaluate the function at the right-hand endpoint of the ith interval to get the height of the ith rectangle, then the area of the ith rectangle will be:

$$\text{base} \cdot \text{height} = \frac{b-a}{n} \cdot f\left(a+i\frac{b-a}{n}\right)$$

Note that if $f\left(a+i\frac{b-a}{n}\right)$ happens to be negative, the area of the ith rectangle will be negative if one uses this formula. Since we want to compute weighted areas, this is actually appropriate. Adding up all of these areas gives us the Right-Hand Rule sum for n rectangles approximating $\displaystyle\int_a^b f(x)\,dx$, namely:

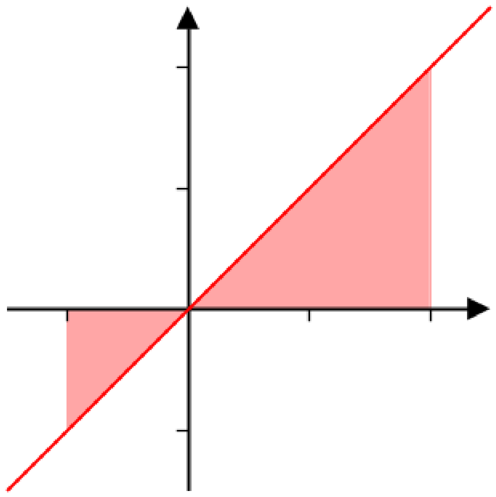
$$\sum_{i=1}^{n} \frac{b-a}{n} \cdot f\left(a+i\cdot\frac{b-a}{n}\right) = \frac{b-a}{n} \cdot f\left(a+1\cdot\frac{b-a}{n}\right) + \cdots + \frac{b-a}{n} \cdot f\left(a+n\cdot\frac{b-a}{n}\right)$$

The Right-Hand Rule for computing the definite integral of $f(x)$, *i.e.* weighted area between $y=f(x)$ and the x-axis, for x between a and b, is what you get when you take the limit of the above sum as $n \to \infty$:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\left[\sum_{i=1}^{n} \frac{b-a}{n} \cdot f\left(a+i\cdot\frac{b-a}{n}\right)\right] = \lim_{n\to\infty}\left[\frac{b-a}{n} \cdot \sum_{i=1}^{n} f\left(a+i\cdot\frac{b-a}{n}\right)\right]$$

This could serve as a definition of the definite integral when $f(x)$ is nice enough, such as when it is continuous on $[a, b]$, but even then some basic properties of definite integrals are hard to prove. As a computational method for calculating definite integrals, it's not partucularly useful because even simple integrals can take a while to work through. (See Section 8.6 in the textbook for a couple of pretty similar, but more likely to be computationally useful, techniques.) Here is a simple example:

Let's try to use the Right-Hand Rule to compute the area between $f(x)=x$ and the x-axis for $-1 \le x \le 2$. The region consists of two triangles, one with base and height 1 below the axis – this is the part for $-1 \le x \le 0$ – and one with base and height 2 above the x-axis – this is the part for $0 \le x \le 2$. This first triangle has area $\frac{1}{2} \cdot \cdot 1 = \frac{1}{2}$, but it



is below the x-axis, so its weighted area is $-\frac{1}{2}$, the second triangle has area $\frac{1}{2} \cdot 2 \cdot 2 = 2$, and since it is above the x-axis, its weighted area is 2. It follows that the weighted area between the graph of the function and the x-axis should be $\int_{-1}^{2} x\,dx = -\frac{1}{2} + 2 = \frac{3}{2} = 1.5$

Let's see if the Right-Hand Rule gives us the same answer. (It better! :-) We plug things into the formula and try to compute the limit. Note that $a = -1$, $b = 2$, and $f(x) = x$ when we plug away. We will make use of the summation formula $\sum_{i=1}^{n} i = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$ along the way.

$$\begin{aligned}
\int_{-1}^{2} x\,dx &= \lim_{n\to\infty} \left[\frac{b-a}{n} \cdot \sum_{i=1}^{n} f\left(a + i \cdot \frac{b-a}{n} \right) \right] \\
&= \lim_{n\to\infty} \left[\frac{2-(1)}{n} \cdot \sum_{i=1}^{n} f\left(-1 + i \cdot \frac{2-(-1)}{n} \right) \right] = \lim_{n\to\infty} \left[\frac{3}{n} \cdot \sum_{i=1}^{n} \left(-1 + i \cdot \frac{3}{n} \right) \right] \\
&= \lim_{n\to\infty} \frac{3}{n} \left[\left(\sum_{i=1}^{n} (-1) \right) + \left(\sum_{i=1}^{n} i \cdot \frac{3}{n} \right) \right] = \lim_{n\to\infty} \frac{3}{n} \left[-n + \frac{3}{n} \sum_{i=1}^{n} i \right] \\
&= \lim_{n\to\infty} \frac{3}{n} \left[-n + \frac{3}{n} \cdot \frac{n(n+1)}{2} \right] = \lim_{n\to\infty} \frac{3}{n} \left[-n + \frac{3}{2}(n+1) \right] = \lim_{n\to\infty} \frac{3}{n} \left[\frac{1}{2}n + \frac{3}{2}) \right] \\
&= \lim_{n\to\infty} \left[\frac{3}{n} \cdot \frac{1}{2}n + \frac{3}{n} \cdot \frac{3}{2}) \right] = \lim_{n\to\infty} \left[\frac{3}{2} + \frac{9}{2n}) \right] = \frac{3}{2} + 0 = \frac{3}{2} = 1.5
\end{aligned}$$

It worked! O frabjous day! Cahoot! Callay!