Mathematics 1110H – Calculus I: Limits, Derivatives, and Integrals
TRENT UNIVERSITY, Fall 2023

Solutions to the Final Examination
15:00-18:00 in SC 137 on Wednesday, 20 December.

Instructions: Do both of parts **X** and **Y**, and, if you wish, part **Z**. Please show all your work, justify all your answers, and simplify these where you reasonably can. When you are asked to do k of n questions, only the first k that are not crossed out will be marked. *If you have a question, or are in doubt about something,* **ask!**

Aids: Any calculator, as long as it can't communicate with other devices; (all sides of) one letter- or A4-size sheet; one brain (no neuron limit).

Part X. Do all four (4) of **1**–**4**.

1. Compute $\dfrac{dy}{dx}$ as best you can in any four (4) of **a**–**f**. *[20 = 4 × 5 each]*

a. $y = e^{x^2+1}$ **b.** $y = xe^{-x}$ **c.** $y = \ln(\cos(x))$

d. $y = \left(x^3+41\right)^{13}$ **e.** $y = \dfrac{x}{1+x^2}$ **f.** $y = \displaystyle\int_{\sqrt{\pi}}^{\cos(x)} t\,dt$

SOLUTIONS. **a.** *Chain Rule and Power Rule.*

$$\frac{dy}{dx} = \frac{d}{dx}e^{x^2+1} = e^{x^2+1}\cdot\frac{d}{dx}\left(x^2+1\right) = e^{x^2+1}\cdot(2x+0) = 2xe^{x^2+1} \quad \square$$

b. *Product Rule and Chain Rule.*

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\left(xe^{-x}\right) = \left[\frac{d}{dx}x\right]e^{-x} + x\left[\frac{d}{dx}e^{-x}\right]\\ &= 1e^{-x} + xe^{-x}\cdot\frac{d}{dx}(-x) = e^{-x} + xe^{-x}(-1) = (1-x)e^{-x} \quad \square\end{aligned}$$

c. *Chain Rule.*

$$\frac{dy}{dx} = \frac{d}{dx}\ln(\cos(x)) = \frac{1}{\cos(x)}\cdot\frac{d}{dx}\cos(x) = \frac{1}{\cos(x)}\cdot(-\sin(x)) = -\tan(x) \quad \square$$

d. *Power Rule and Chain Rule.*

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\left(x^3+41\right)^{13} = 13\left(x^3+41\right)^{12}\frac{d}{dx}\left(x^3+41\right)\\ &= 13\left(x^3+41\right)^{12}\left(3x^2+0\right) = 39x^2\left(x^3+41\right)^{12} \quad \square\end{aligned}$$

e. *Quotient Rule and Power Rule.*

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\left(\frac{x}{1+x^2}\right) = \frac{\left[\frac{d}{dx}x\right]\left(1+x^2\right) - x\left[\frac{d}{dx}\left(1+x^2\right)\right]}{\left(1+x^2\right)^2}\\ &= \frac{[1]\left(1+x^2\right) - x[0+2x]}{\left(1+x^2\right)^2} = \frac{1+x^2-2x^2}{\left(1+x^2\right)^2} = \frac{1-x^2}{\left(1+x^2\right)^2} \quad \square\end{aligned}$$

f. *Integrate, then Power Rule and Chain Rule.*

$$\frac{dy}{dx} = \frac{d}{dx}\left[\int_{\sqrt{\pi}}^{\cos(x)} t\,dt\right] = \frac{d}{dx}\left[\left.\frac{t^2}{2}\right|_{\sqrt{\pi}}^{\cos(x)}\right] = \frac{d}{dx}\left[\frac{\cos^2(x)}{2} - \frac{\pi}{2}\right]$$
$$= \frac{1}{2}\cdot 2\cos(x)\cdot\frac{d}{dx}\cos(x) - 0 = \cos(x)\left(-\sin(x)\right) = -\cos(x)\sin(x) = -\frac{1}{2}\sin(2x) \quad \square$$

f. *Fundamental Theorem of Calculus and Chain Rule.*

$$\frac{dy}{dx} = \frac{d}{dx}\left[\int_{\sqrt{\pi}}^{\cos(x)} t\,dt\right] = \left[\frac{d}{du}\int_{\sqrt{\pi}}^{u} t\,dt\right]\cdot\frac{du}{dx} \quad [\text{where } u = \cos(x)] \quad = u\cdot\frac{du}{dx}$$
$$= \cos(x)\cdot\frac{d}{dx}\cos(x) = \cos(x)\left(-\sin(x)\right) = -\cos(x)\sin(x) = -\frac{1}{2}\sin(2x) \quad \blacksquare$$

2. Evaluate any four (4) of the integrals **a**–**f**. *[20 = 4 × 5 each]*

a. $\displaystyle\int_0^1 \frac{x+1}{x^2+2x+1}\,dx$ **b.** $\displaystyle\int x\sec^2(x)\,dx$ **c.** $\displaystyle\int \tanh(x)\,dx$

d. $\displaystyle\int_0^{\pi/2} \sin(2x)\,dx$ **e.** $\displaystyle\int_0^1 \frac{x^2}{e^x}\,dx$ **f.** $\displaystyle\int \frac{\arctan(x)}{1+x^2}\,dx$

SOLUTIONS. **a.** *Simplification and Substitution.*

$$\int_0^1 \frac{x+1}{x^2+2x+1}\,dx = \int_0^1 \frac{x+1}{(x+1)^2}\,dx = \int_0^1 \frac{1}{x+1}\,dx \quad \begin{array}{l}\text{Let } w = x+1, \text{ so } dw = dx,\\ \text{and change the limits: } \begin{smallmatrix} x & 0 & 1\\ w & 1 & 2\end{smallmatrix}\end{array}$$
$$= \int_1^2 \frac{1}{w}\,dw = \left.\ln(w)\right|_1^2 = \ln(2) - \ln(1) = \ln(2) - 0 = \ln(2) \quad \square$$

a. *Substitution.* We will substitute right away, with $u = x^2+2x+1$, so $du = (2x+2)\,dx$ and $(x+1)\,dx = \frac{1}{2}\,du$. This time we will keep the old limits and substitute back before using them.

$$\int_0^1 \frac{x+1}{x^2+2x+1}\,dx = \int_{x=0}^{x=1} \frac{1}{u}\cdot\frac{1}{2}\,du = \left.\frac{1}{2}\ln(u)\right|_{x=0}^{x=1} = \left.\frac{1}{2}\ln\left(x^2+2x+1\right)\right|_0^1$$
$$= \frac{1}{2}\ln(4) - \frac{1}{2}\ln(1) = \frac{1}{2}\ln\left(2^2\right) - \frac{1}{2}\cdot 0 = \frac{1}{2}\cdot 2\ln(2) = \ln(2) \quad \square$$

b. *Integration by Parts and Substitution.* We will use the parts $u = x$ and $v' = \sec^2(x)$, so $u' = 1$ and $v = \tan(x)$. We will then integrate $\tan(x) = \dfrac{\sin(x)}{\cos(x)}$ using the substitution $w = \cos(x)$, so $dw = -\sin(x)\,dx$ and $\sin(x)\,dx = (-1)\,dw$.

$$\int x\sec^2(x)\,dx = x\tan(x) - \int 1\tan(x)dx = x\tan(x) - \int \frac{\sin(x)}{\cos(x)}\,dx$$
$$= x\tan(x) - \int \frac{1}{w}\cdot(-1)\,dw = x\tan(x) + \int \frac{1}{w}\,dw$$
$$= x\tan(x) + \ln(w) + C = x\tan(x) + \ln\left(\cos(x)\right) + C \quad \square$$

c. *Substitution.* Recall that the hyperbolic function $\tanh(x) = \dfrac{\sinh(x)}{\cosh(x)}$ and also that $\dfrac{d}{dx}\cosh(x) = \sinh(x)$. (No negative sign here, unlike with the trigonometric counterparts.) We will use the substitution $u = \cosh(x)$, so $du = \sinh(x)\,dx$.

$$\int \tanh(x)\,dx = \int \frac{\sinh(x)}{\cosh(x)}\,dx = \int \frac{1}{u}\,du = \ln(u) + C = \ln\left(\cosh(x)\right) + C \quad \square$$

d. *Substitution.* We will use the substitution $z = 2x$, so $dz = 2\,dx$ and $dx = \frac{1}{2}\,dz$, changing the limits as we go along: $\begin{matrix} x & 0 & \pi/2 \\ z & 0 & \pi \end{matrix}$

$$\begin{aligned}\int_0^{\pi/2} \sin(2x)\,dx &= \int_0^{\pi} \sin(z)\cdot\frac{1}{2}\,dz = -\frac{1}{2}\cos(z)\Big|_0^{\pi} = \left(-\frac{1}{2}\cos(\pi)\right) - \left(-\frac{1}{2}\cos(0)\right) \\ &= -\frac{1}{2}(-1) - \left(-\frac{1}{2}\cdot 1\right) = \frac{1}{2} + \frac{1}{2} = 1 \quad \square\end{aligned}$$

NOTE. One could also do **d** by using the identity $\sin(2x) = 2\sin(x)\cos(x)$ and then substituting for $\sin(x)$ or $\cos(x)$, but that's a little more work.

e. *Integration by Parts, twice over.* We will also use the fact that $\dfrac{1}{e^x} = e^{-x}$. Note that $\int e^{-x}\,dx = -e^{-x}$ if one substitutes for $-x$.

$$\begin{aligned}\int_0^1 \frac{x^2}{e^x}\,dx &= \int_0^1 x^2 e^{-x}\,dx \qquad \begin{array}{l}\text{Let } u = x^2 \text{ and } v' = e^{-x}\text{, so} \\ u' = 2x \text{ and } v = -e^{-x}.\end{array} \\ &= -x^2e^{-x}\Big|_0^1 - \int_0^1 2x\left(-e^{-x}\right)\,dx \\ &= -x^2e^{-x}\Big|_0^1 + 2\int_0^1 xe^{-x}\,dx \qquad \begin{array}{l}\text{Let } s = x \text{ and } t' = e^{-x}\text{, so} \\ s' = 1 \text{ and } t = -e^{-x}.\end{array} \\ &= -x^2e^{-x}\Big|_0^1 + 2\left[-xe^{-x}\Big|_0^1 - \int_0^1 1\left(-e^{-x}\right)\,dx\right] \\ &= \left(-1^2e^{-1}\right) - \left(-0^2e{-0}\right) + 2\left[-xe^{-x}\Big|_0^1 + \int_0^1 e^{-x}\,dx\right] \\ &= -\frac{1}{e} - 0 + 2\left[\left(-1e^{-1}\right) - \left(-0e^{-0}\right) + \left(-e^{-x}\right)\Big|_0^1\right] \\ &= -\frac{1}{e} + 2\left[-\frac{1}{e} - 0 + \left(-e^{-1}\right) - \left(-e^{-0}\right)\right] \\ &= -\frac{1}{e} + 2\left[-\frac{1}{e} - \frac{1}{e} + 1\right] = -\frac{1}{e} - \frac{4}{e} + 2 = 2 - \frac{5}{e} \quad \square\end{aligned}$$

f. *Substitution.* We will use the substitution $w = \arctan x$, so $dw = \dfrac{1}{1+x^2}\,dx$.

$$\int \frac{\arctan(x)}{1+x^2}\,dx = \int w\,dw = \frac{w^2}{2} + C = \frac{1}{2}\arctan^2(x) + C \quad \blacksquare$$

3. Do any four (4) of **a**–**f**. *[20 = 4 × 5 each]*

a. Use the ε–δ definition of limits to verify that $\lim_{x\to 1}(3x-5) = -2$.

b. At what value(s) of x, if any, does the graph of $y = \dfrac{x}{1+x^2}$ have a tangent line with slope 1?

c. Compute $\displaystyle\lim_{x\to\infty} \frac{x^2+1}{e^x+1}$.

d. Find $g(x)$ if $g'(x) = \cos(\pi x)$ and $g(1) = \dfrac{1}{\pi}$.

e. Let $h(x) = \begin{cases} e^{-x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$. Determine whether $h(x)$ is continuous at $x = 0$.

f. Sketch the finite region between $y = x+1$ and $y = 2^x$, for $0 \leq x \leq 1$, and find its area.

SOLUTIONS. **a.** We need to show that for every $\varepsilon > 0$ there is a $\delta > 0$ such that if $|x-1| < \delta$, then $|(3x-5)-(-2)| < \varepsilon$. As usual, we attempt to reverse-engineer the required δ from $|(3x-5)-(-2)| < \varepsilon$.

Suppose we are given an $\varepsilon > 0$. Then:

$$\begin{aligned} |(3x-5)-(-2)| < \varepsilon &\iff |3x-5+2| < \varepsilon \iff |3x-3| < \varepsilon \\ &\iff 3|x-1| < \varepsilon \iff |x-1| < \frac{\varepsilon}{3}, \end{aligned}$$

Since every step above is reversible, if we set $\delta = \dfrac{\varepsilon}{3}$, we will have that if $|x-1| < \delta$, then $|(3x-5)-(-2)| < \varepsilon$, as required. It follows that $\lim_{x\to 1}(3x-5) = -2$ by the ε–δ definition of limits. $\square$

b. The slope of the tangent line is given by the derivative of the function. We could cite **1e** if we did it, otherwise we have:

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}\left(\frac{x}{1+x^2}\right) = \frac{\left[\frac{d}{dx}x\right]\left(1+x^2\right) - x\left[\frac{d}{dx}\left(1+x^2\right)\right]}{\left(1+x^2\right)^2} \\ &= \frac{[1]\left(1+x^2\right) - x\,[0+2x]}{\left(1+x^2\right)^2} = \frac{1+x^2-2x^2}{\left(1+x^2\right)^2} = \frac{1-x^2}{\left(1+x^2\right)^2} \end{aligned}$$

The derivative is defined for all x because $1+x^2 \geq 1 > 0$ for all x. So when is the slope 1?

$$\begin{aligned} \frac{dy}{dx} = 1 &\iff \frac{1-x^2}{\left(1+x^2\right)^2} = 1 \iff 1-x^2 = \left(1+x^2\right)^2 = 1+2x^2+x^4 \\ &\iff x^4+3x^2 = 0 \iff x^2\left(x^2+3\right) = 0 \iff x^2 = 0 \text{ or } x^2 = -3 \\ &\iff x = 0 \text{ or } \cancel{x = \sqrt{-3}} \end{aligned}$$

Thus the graph of $y = \dfrac{x}{1+x^2}$ has a tangent line with slope 1 exactly when $x = 0$. $\square$

c. *Informal.* Since the natural exponential function e^x eventually dominates any polynomial function, such as x^2+1, as x increases, $\lim_{x\to\infty} \frac{x^2+1}{e^x+1} = 0$. $\square$

c. *At least semi-formal.* We will use l'Hôpital's Rule twice to compute the limit:

$$\begin{aligned}
\lim_{x\to\infty} \frac{x^2+1 \to \infty}{e^x+1 \to \infty} &= \lim_{x\to\infty} \frac{\frac{d}{dx}\left(x^2+1\right)}{\frac{d}{dx}\left(e^x+1\right)} = \lim_{x\to\infty} \frac{2x \to \infty}{e^x \to \infty} \\
&= \lim_{x\to\infty} \frac{\frac{d}{dx}(2x)}{\frac{d}{dx}e^x} = \lim_{x\to\infty} \frac{2 \to 2}{e^x \to \infty} = 0 \quad \square
\end{aligned}$$

d. If $g'(x) = \cos(\pi x)$, then $g(x)$, the anti-derivative of $g'(x)$ is given by:

$$\begin{aligned}
g(x) &= \int g'(x)\,dx = \int \cos(\pi x)\,dx \qquad \text{Substitute } u = \pi x \text{, so } du = \pi\,dx \text{ and } dx = \tfrac{1}{\pi}\,du. \\
&= \int \cos(u) \cdot \frac{1}{\pi}\,du = \frac{1}{\pi}\sin(u) + C = \frac{1}{\pi}\sin(\pi x) + C
\end{aligned}$$

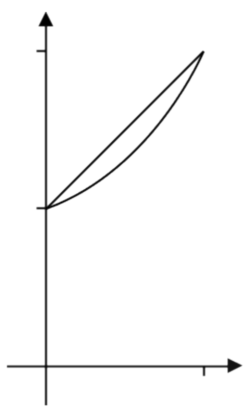
It remains to determine the value of the constant C. We are given that $g(1) = \frac{1}{\pi}$, so $\frac{1}{\pi} = g(1) = \frac{1}{\pi}\sin(\pi 1) + C = \frac{1}{\pi}\sin(\pi) + C = \frac{1}{\pi}0 + C = 0 + C = C$. Thus $g(x) = \frac{1}{\pi}\sin(\pi x) + \frac{1}{\pi} = \frac{\sin(\pi x)+1}{\pi}$. $\square$

e. By the definition of continuity, $h(x)$ is continuous at 0 if $\lim_{x\to 0} h(x)$ exists and is equal to $h(0)$. Since $h(x) = \begin{cases} e^{-x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$, this boils down to checking whether $\lim_{x\to 0} e^{-x^2} = 0$:

$$\lim_{x\to 0} e^{-x^2} = \lim_{x\to 0} \frac{1}{e^{x^2}} = \frac{1}{e^{0^2}} = \frac{1}{e^0} = \frac{1}{1} = 1 \neq 0$$

Since $\lim_{x\to 0} h(x) \neq h(0)$, $h(x)$ is not continuous at $x = 0$. $\square$

f. Here is a sketch of the region:



Note that the graphs intersect at $x = 0$, since $0 + 1 = 1 = 2^0$, and at $x = 1$, since $1 + 1 = 2 = 2^1$. For x between 0 and 1, $y = x + 1$ is above $y = 2^x$; for example, at $x = \frac{1}{2}$ we have $2^{1/2} = \sqrt{2} \approx 1.4142 < 1.5 = \frac{3}{2} = \frac{1}{2} + 1$. It follows that the area of the finite region between $y = x + 1$ and $y = 2^x$, for $0 \leq x \leq 1$, is:

$$\begin{aligned}
\text{Area} &= \int_0^1 (x + 1 - 2^x)\, dx = \int_0^1 (x+1)\, dx - \int_0^1 2^x\, dx = \left(\frac{x^2}{2} + x \right)\Big|_0^1 - \frac{2^x}{\ln(2)}\Big|_0^1 \\
&= \left[\left(\frac{1^2}{2} + 1 \right) - \left(\frac{0^2}{2} + 0 \right) \right] - \left[\frac{2^1}{\ln(2)} - \frac{2^0}{\ln(2)} \right] = \left[\frac{3}{2} - 0 \right] - \left[\frac{2}{\ln(2)} - \frac{1}{\ln(2)} \right] \\
&= \frac{3}{2} - \frac{1}{\ln(2)} \approx 1.5 - 1.4427 \approx 0.0573 \quad \blacksquare
\end{aligned}$$

4. Find the domain, intercepts, vertical and horizontal asymptotes, intervals of increase and decrease, maximum and minimum points, intervals of concavity, and inflection points of $f(x) = \dfrac{e^x}{e^x + 1}$. *[12]*

SOLUTION. We run through the indicated checklist:

i. Domain. Since $e^x > 0$ for all x, $1 + e^x \neq 0$ for all x. It follows that $f(x) = \dfrac{e^x}{e^x + 1}$ is defined (and continuous and differentiable) for all x, so its domain is all x, or $\mathbb{R}$, or $(-\infty, \infty)$, or ...

ii. Intercepts. $f(0) = \dfrac{e^0}{e^0 + 1} = \dfrac{1}{1+1} = \dfrac{1}{2}$, so $f(x)$ has y-intercept $\frac{1}{2}$. Since the numerator e^x is > 0 for all x, $f(x) \neq 0$ for all x, so there is no x-intercept.

iii. Vertical Asymptotes. Since $f(x)$ is defined and continuous for all x, it cannot have any vertical asymptote.

iv. Horizontal Asymptotes. We take the limits as $x \to \pm\infty$ and see what happens:

$$\begin{aligned}
\lim_{x \to -\infty} f(x) &= \lim_{x \to -\infty} \frac{e^x \to 0^+}{e^x + 1 \to 0^+ + 1} = \frac{0^+}{1^+} = 0^+ \\
\lim_{x \to \infty} f(x) &= \lim_{x \to \infty} \frac{e^x}{e^x + 1} = \lim_{x \to \infty} \frac{e^x}{e^x + 1} \cdot \frac{1/e^x}{1/e^x} \\
&= \lim_{x \to \infty} \frac{1 \to 1}{1 + 1/e^x \to 1 + 0^+} = \frac{1}{1 + 0^+} = 1^-
\end{aligned}$$

Note that $f(x)$ approaches 0 from above as $x \to -\infty$ since both the numerator and denominator are positive, and that $f(x)$ approaches 1 from below as $x \to \infty$ since both the numerator and denominator are positive and the numerator is always smaller than the denominator. It follows that $f(x)$ has horizontal asymptotes of $y = 0$ as $x \to -\infty$ and $y = 1$ as $x \to \infty$.

v. Increase/Decrease and Maxima/Minima. We compute the first derivative, using the Quotient Rule:

$$f'(x) = \frac{d}{dx}\left(\frac{e^x}{e^x+1}\right) = \frac{\left[\frac{d}{dx}e^x\right](e^x+1) - e^x\left[\frac{d}{dx}(e^x+1)\right]}{(e^x+1)^2}$$
$$= \frac{[e^x](e^x+1) - e^x[e^x+0]}{(e^x+1)^2} = \frac{(e^x)^2 + e^x - (e^x)^2}{(e^x+1)^2} = \frac{e^x}{(e^x+1)^2}$$

Since $e^x > 0$ and $(e^x+1)^2 > 0$ for all x, $f'(x)$ is defined and continuous and differentiable for all x, and also $f'(x) > 0$ for all x. $f(x)$ therefore has no critical points and is always increasing. We summarize this in the usual (pretty trivial in this case) table:

x	$(-\infty, \infty)$
$f'(x)$	$+$
$f(x)$	$\uparrow$

Note, in particular, that $f(x)$ has no maximum or minimum points.

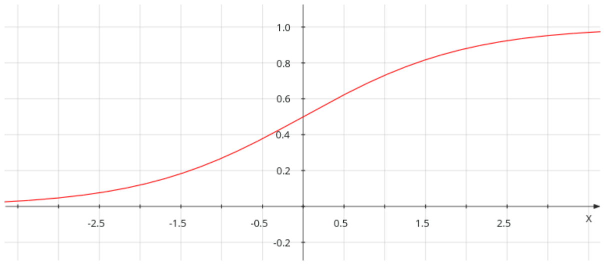
vi. Concavity and Inflection. We compute the second derivative, using the Quotient Rule and the Chain Rule:

$$f''(x) = \frac{d}{dx}\left(\frac{e^x}{(e^x+1)^2}\right) = \frac{\left[\frac{d}{dx}e^x\right](e^x+1)^2 - e^x\left[\frac{d}{dx}(e^x+1)^2\right]}{\left((e^x+1)^2\right)^2}$$
$$= \frac{[e^x](e^x+1)^2 - e^x\left[2(e^x+1)\cdot\frac{d}{dx}(e^x+1)\right]}{(e^x+1)^4}$$
$$= \frac{e^x(e^x+1)^2 - e^x[2(e^x+1)e^x]}{(e^x+1)^4} = \frac{e^x(e^x+1)^2 - 2(e^x)^2(e^x+1)}{(e^x+1)^4}$$
$$= \frac{e^x(e^x+1) - 2(e^x)^2}{(e^x+1)^3} = \frac{(e^x)^2 + e^x - 2(e^x)^2}{(e^x+1)^3} = \frac{+e^x - (e^x)^2}{(e^x+1)^3}$$
$$= \frac{e^x(1-e^x)}{(e^x+1)^3}$$

Since $e^x + 1 > 0$ for all x, $f''(x)$ is defined and continuous and differentiable for all x. As $e^x > 0$ and $(e^x+1)^3 > 0$ for all x, $f''(x)$ is positive, negative, or zero exactly as $1 - e^x$ is. $1 - e^x = 0 \iff e^x = 1 \iff x = 0$. Similarly, $1 - e^x < 0 \iff e^x > 1 \iff x > 0$ and $1 - e^x > 0 \iff e^x < 1 \iff x < 0$. Thus $f(x)$ is concave up when $x < 0$ and concave down when $x > 0$, and so has an inflection point at $x = 0$. We summarize this information in the usual table:

x	$(-\infty, 0)$	0	$(0, \infty)$
$f''(x)$	$+$	0	$-$
$f(x)$	$\smile$	infl	$\frown$

vii. The Graph. It wasn't actually asked for, and it might be cheating a little to have a computer do the work, but here is a piece of the graph:



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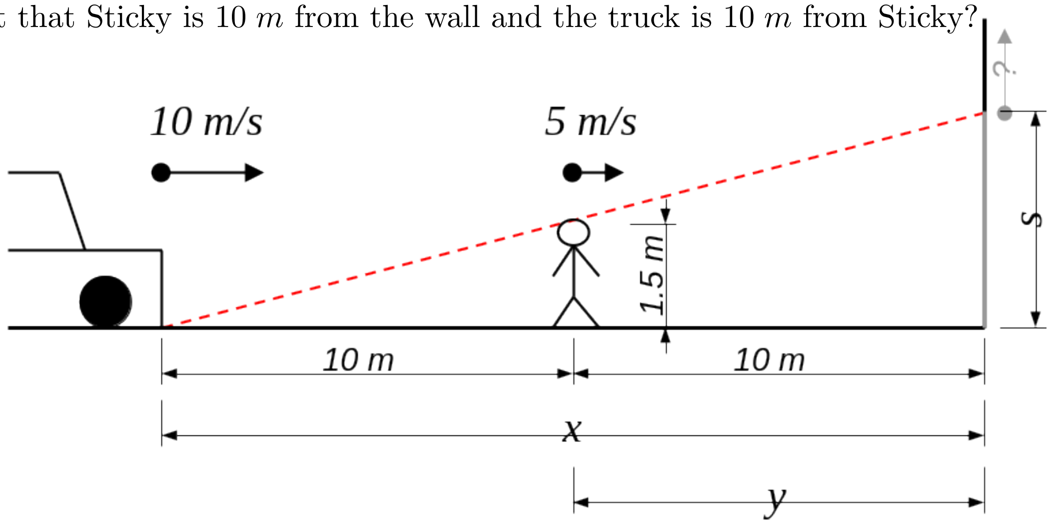
Part Y. Do any two (2) of **5**–**7**. *[28 = 2 × 14 each]* ... *and here is "more"!*

5. Sticky is trapped in a dead-end alley! An extreme low-rider truck with its headlights at ground level is chasing sticky down the alley, straight towards the wall at the end of the alley. Sticky, who is 1.5 m tall, is running towards the wall at a constant speed of 5 m/s and the truck is driving towards Sticky and the wall at a constant speed of 10 m/s. Sticky casts a shadow in the light from the truck headlights upon the wall at the end of the alley. How is the tip of Sticky's shadow moving on the wall at the instant that Sticky is 10 m from the wall and the truck is 10 m from Sticky?



SOLUTION. Let x, y, and s be as indicated in the annotated diagram above. That is, let x be the distance between the truck and the wall, so we are given that $\frac{dx}{dt} = -10\ m/s$ (the sign is negative because the distance is diminishing) and at the instant in question we have that $x = 10 + 10 = 20\ m$. Similarly, let y be the distance between Sticky and the wall, so we are given that $\frac{dy}{dt} = -5\ m/s$ (again, the sign is negative because the distance is diminishing) and at the instant in question we have that $y = 10\ m$. Finally, let s be the height (or length on the wall) of the shadow cast by Sticky in the truck's headlights; our task is to determine $\frac{ds}{dt}$.

Consider the triangle formed by the headlight beam, Sticky, and the ground, as well as the triangle formed by the headlight beam, the shadow on the wall, and the ground. Both are right triangles, with the hypotenuse formed by the headlight beam, and both share a common angle, that between the headlight beam and the ground. Since they have two corresponding angles that are equal, it follows that these two triangles are similar, and hence that corresponding sides have the same ratios. Since the vertical sides are 1.5 m and s m for the smaller and larger triangles respectively, and the horizontal sides are $x - y$ m and x m respectively, it follows that $\frac{s}{x} = \frac{1.5}{x-y}$.

It follows in turn that $s = \frac{1.5x}{x-y}$, and so, with the help of the Quotient Rule, $\frac{ds}{dt}$ at any giiven instant is given by:

$$\begin{aligned}
\frac{ds}{dt} &= \frac{d}{dt}\left(\frac{1.5x}{x-y}\right) = 1.5\frac{d}{dt}\left(\frac{x}{x-y}\right) = 1.5\frac{\left[\frac{dx}{dt}\right](x-y) - x\left[\frac{d}{dt}(x-y)\right]}{(x-y)^2} \\
&= 1.5\frac{[-10](x-y) - x\left[\frac{dx}{dt} - \frac{dy}{dt}\right]}{(x-y)^2} = 1.5\frac{-10(x-y) - x[-10-(-5)]}{(x-y)^2} \\
&= 1.5\frac{-10x + 10y - x[-5]}{(x-y)^2} = 1.5\frac{-10x + 10y + 5x}{(x-y)^2} = 1.5\frac{10y - 5x}{(x-y)^2}
\end{aligned}$$

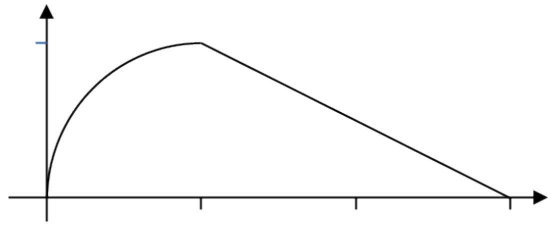
At the instant that Sticky is 10 m from the wall and the truck is 10 m from Sticky, we have $x = 20$ and $y = 10$, so at this instant we also have

$$\frac{ds}{dt} = 1.5\frac{10 \cdot 10 - 5 \cdot 20}{(20-10)^2} = 1.5\frac{100-100}{10^2} = 1.5\frac{0}{100} = 0\ m/s.$$

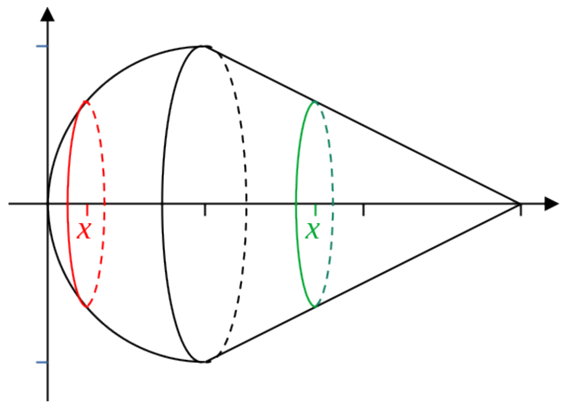
That is, at this instant the length/height of shadow is not changing at all, so its tip is not moving. ■

Note: In this setup the shadow will not be changing at all until the truck meets the wall with Sticky between them. Work out why!

6. The region between the x-axis and $y = \sqrt{x}$, for $0 \le x \le 1$, and between the x-axis and $y = -\frac{x}{2} + \frac{3}{2}$, for $1 \le x \le 3$, is revolved about the x-axis. Sketch the resulting solid and find its volume.



SOLUTION. Here is a sketch of the solid:



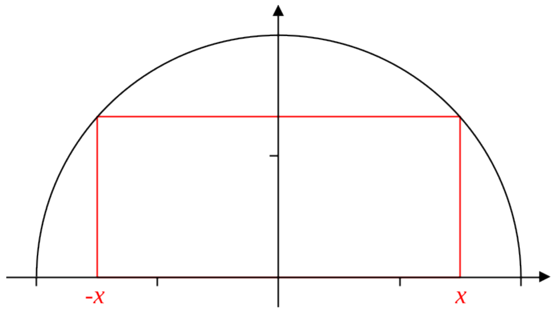
We will use the disk/washer method to compute the volume of this "ice-cream cone". When $0 \le x \le 1$, the cross-sectional disk at x has radius $r = y - 0 = \sqrt{x}$ and hence area $A(x) = \pi r^2 = \pi \left(\sqrt{x}\right)^2 = \pi x$, and when $1 \le x \le 3$, the disk at x has radius $r = y - 0 = -\frac{1}{2}x + \frac{3}{2}$ and hence area $A(x) = \pi r^2 = \pi \left(-\frac{1}{2}x + \frac{3}{2}\right)^2 = \pi \left(\frac{1}{4}x^2 - \frac{3}{2}x + \frac{9}{4}\right)$. It follows that the volume of the ice-cream cone is:

$$
\begin{aligned}
V &= \int_0^3 A(x)\,dx = \int_0^1 A(x)\,dx + \int_1^3 A(x)\,dx = \int_0^1 \pi x\,dx + \int_1^3 \pi\left(\frac{1}{4}x^2 - \frac{3}{2}x + \frac{9}{4}\right)dx \\
&= \left.\frac{\pi x^2}{2}\right|_0^1 + \pi\left.\left(\frac{1}{4}\cdot\frac{x^3}{3} - \frac{3}{2}\cdot\frac{x^2}{2} + \frac{9}{4}\cdot x\right)\right|_1^3 = \frac{\pi 1^2}{2} - \frac{\pi 0^2}{2} + \pi\left.\left(\frac{x^3}{12} - \frac{3x^2}{4} + \frac{9x}{4}\right)\right|_1^3 \\
&= \frac{\pi}{2} - 0 + \pi\left(\frac{3^3}{12} - \frac{3\cdot 3^2}{4} + \frac{9\cdot 3}{4}\right) - \pi\left(\frac{1^3}{12} - \frac{3\cdot 1^2}{4} + \frac{9\cdot 1}{4}\right) \\
&= \frac{\pi}{2} + \pi\left(\frac{27}{12} - \frac{27}{4} + \frac{27}{4}\right) - \pi\left(\frac{1}{12} - \frac{3}{4} + \frac{9}{4}\right) \\
&= \frac{6\pi}{12} + \pi\cdot\frac{27}{12} - \pi\cdot\frac{19}{12} = \frac{14\pi}{12} = \frac{7\pi}{6}
\end{aligned}
$$

Thus the volume of this "ice=cream cone" is $\frac{7\pi}{6}$. [Units, you ask? What's that? :-)] ■

NOTE. One could also do this problem, with overall about an equal amount of work, using the cylindrical shell method.

7. Find the maximum possible area of a rectangle whose base is on the x-axis and whose upper corners are on the semi-circle $y = \sqrt{4 - x^2}$.



SOLUTION. The semi-circle $y = \sqrt{4-x^2}$ is symmetric about the y-axis because $\sqrt{4-x^2} = \sqrt{4-(-x)^2}$. It follows that a rectangle whose base is on the x-axis, with one corner at x, and whose upper corners are on the semi-circle $y = \sqrt{4-x^2}$, will have its other corner on the x-axis be at $-x$. (x and $-x$ added above to the originally given diagram.) Note that the semi-circle meets the x-axis at $x = \pm 2$, and that $\sqrt{4-x^2}$ is undefined when $x < -2$ or $x > 2$. It follows that we only to deal with $-2 \le x \le 2$ in this problem. In fact, since x and $-x$ both give the same rectangle, we can stick to $0 \le x \le 2$.

The base of the rectangle with a corner at x, where $0 \le x \le 2$, has length $x-(-x) = 2x$ and its height is $y - 0 = y = \sqrt{4-x^2}$, so its area is $A(x) = \text{base} \cdot \text{height} = 2x\sqrt{4-x^2}$. We need to maximize this area function on the interval $[0, 2]$; note that $A(x)$ is continuous on this interval, so we can do so by comparing the value of $A(x)$ at the endpoints with its value at any critical points in the interval.

As usual, we first look for critical points, where the derivative $A'(x)$ is undefined or zero. We compute said derivative with the help of the Product, Chain, and Power Rules:

$$\begin{aligned}
A'(x) &= \frac{d}{dx}\left(2x\sqrt{4-x^2}\right) = \frac{d}{dx}\left(2x\left(4-x^2\right)^{1/2}\right) \\
&= \left[\frac{d}{dx}(2x)\right]\left(4-x^2\right)^{1/2} + 2x \cdot \left[\frac{d}{dx}\left(4-x^2\right)^{1/2}\right] \\
&= 2\left(4-x^2\right)^{1/2} + 2x \cdot \frac{1}{2}\left(4-x^2\right)^{-1/2}\left[\frac{d}{dx}\left(4-x^2\right)\right] \\
&= 2\left(4-x^2\right)^{1/2} + x\left(4-x^2\right)^{-1/2} \cdot (-2x) \\
&= 2\sqrt{4-x^2} + \frac{-2x^2}{\sqrt{4-x^2}} = \frac{2\left(\sqrt{4-x^2}\right)^2}{\sqrt{4-x^2}} - \frac{2x^2}{\sqrt{4-x^2}} \\
&= \frac{2\left(4-x^2\right) - 2x^2}{\sqrt{4-x^2}} = \frac{8-2x^2-2x^2}{\sqrt{4-x^2}} = \frac{8-4x^2}{\sqrt{4-x^2}}
\end{aligned}$$

Note that the derivative is defined for all x with $-2 < x < 2$; in particular, it is defined at all points in the interval $[0, 2]$ except the right-hand endpoint, which we were going to check separately anyway. Looking for critical points, observe that $A'(x) = 0$ exactly when its numerator is 0, so:

$$A'(x) = 0 \iff 8 - 4x^2 = 0 \iff 4x^2 = 8 \iff x^2 = 2 \iff x = \pm\sqrt{2}$$

$x = -\sqrt{2}$ is not in the interval $[0, 2]$, but $x = \sqrt{2}$ is.

We compare the values of $A(x) = 2x\sqrt{4 - x^2}$ at the endpoints $x = 0$ and $x = 2$ with its value at the critical point $x = \sqrt{2}$ to find the maximum.

$$\begin{aligned} A(0) &= 2 \cdot 0 \cdot \sqrt{4 - 0^2} = 0 \\ A(2) &= 2 \cdot 2 \cdot \sqrt{4 - 2^2} = 4 \cdot \sqrt{0} = 0 \\ A\left(\sqrt{2}\right) &= 2 \cdot \sqrt{2} \cdot \sqrt{4 - \left(\sqrt{2}\right)^2} = 2\sqrt{2} \cdot \sqrt{4 - 2} = 2\sqrt{2} \cdot \sqrt{2} = 2 \cdot 2 = 4 \end{aligned}$$

Thus the maximum possible area of a rectangle whose base is on the x-axis and whose upper corners are on the semi-circle $y = \sqrt{4 - x^2}$ is 4, which occurs when $x = \sqrt{2}$. ∎

[Total = 100]

Part Z. *Here be bonus points!* Do one or both of $\mathbf{7+1}$ and $\mathbf{7+2}$.

$\mathbf{7+1}$. Give a funny and clever way for Sticky to get out of the sticky situation described in question **5**. *[1]*

Solution. Please help Sticky! :-) ∎

$\mathbf{7+2}$. Does Euler's polynomial $x^2 - x + 41$ always give you a prime number when x is a positive integer? If it does, explain why; if not, give an example where it doesn't. *[1]*

Solution. It doesn't, since $41^2 - 41 + 41 = 41^2$, which is not a prime number since it has a positive integer factor, namely 41, other than itself and 1.

The really remarkable thing is that for every positive integer x with $1 \leq x \leq 40$, $x^2 - x + 41$ is indeed a prime number. ∎

I hope that you enjoyed the course. Enjoy the break even more!