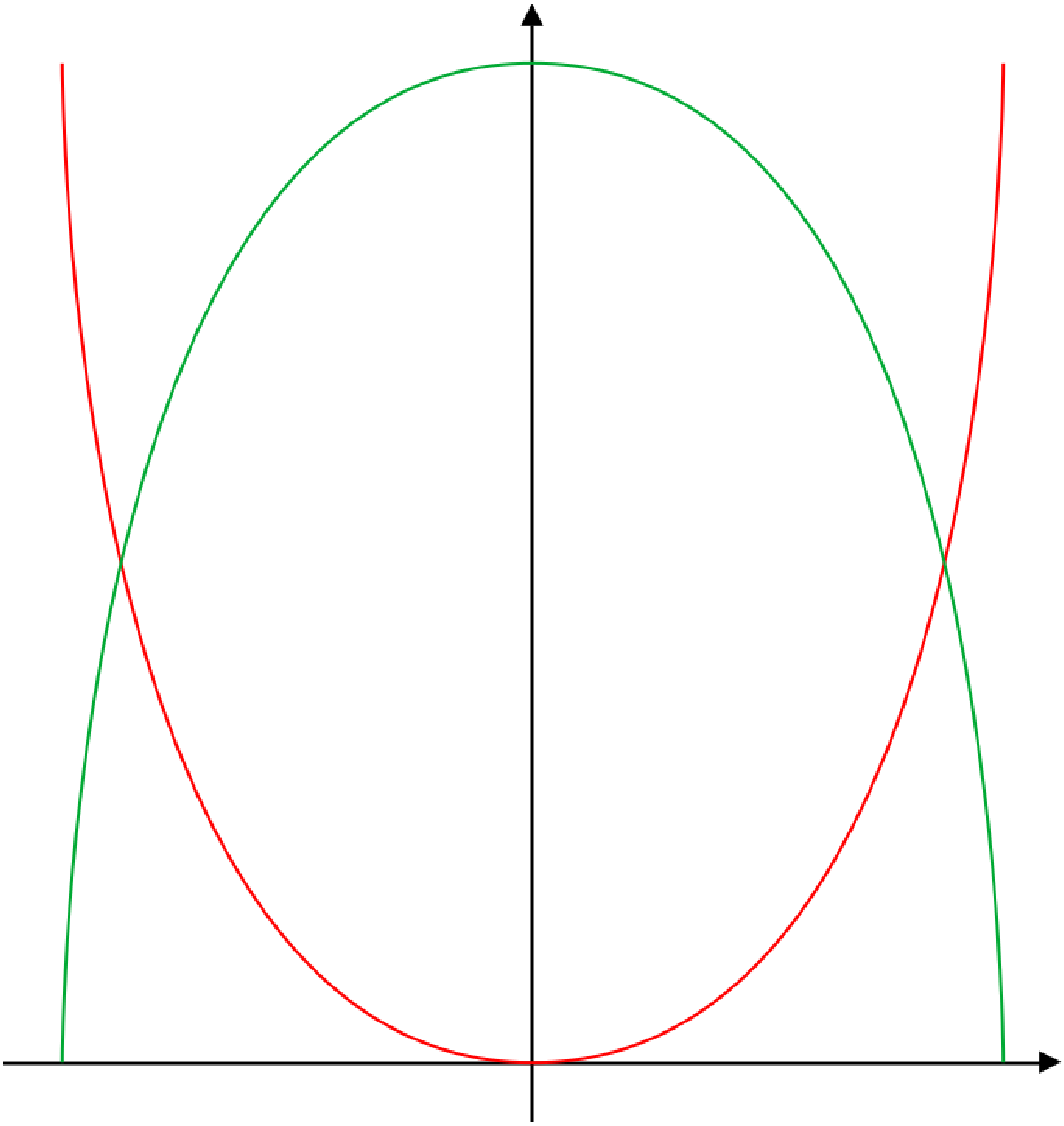
Mathematics 1110H – Calculus I: Limits, Derivatives, and Integrals
Trent University, Fall 2023

Solutions to Assignment #6
An Area Problem



1. Find the real number $a > 0$ such that the area of the finite region below the parabola $y = a - x^2$ and above the parabola $y = x^2$ is exactly $\frac{64}{3}$. *[10]*

Hint: This could be done entirely by hand. If you'd rather not, though, it is worth noting that SageMath can integrate functions, both symbolically and numerically, and solve various kinds of equations.

Solution 1. *(Entirely by hand.)* The two parabolas $y = x^2$ and $y = a - x^2$ intersect when $x^2 = a - x^2$. Rearranging just a bit, this means that $2x^2 = a$, so $x^2 = a/2$, and thus $x = \pm\sqrt{a/2}$. It follows that the area of the finite region between the two parabolas is therefore:

$$
\begin{aligned}
\text{Area} &= \int_{-\sqrt{a/2}}^{\sqrt{a/2}} (\text{upper} - \text{lower})\ dx = \int_{-\sqrt{a/2}}^{\sqrt{a/2}} \left(\left(a - x^2\right) - x^2\right)\ dx \\
. &= \int_{-\sqrt{a/2}}^{\sqrt{a/2}} \left(a - 2x^2\right)\ dx = \left(ax - \frac{2x^3}{3}\right)\Bigg|_{-\sqrt{a/2}}^{\sqrt{a/2}} \\
&= \left(a\sqrt{a/2} - \frac{2\left(\sqrt{a/2}\right)^3}{3}\right) - \left(a\left(-\sqrt{a/2}\right) - \frac{2\left(-\sqrt{a/2}\right)^3}{3}\right) \\
&= \left(\frac{a^{3/2}}{\sqrt{2}} - \frac{2a^{3/2}}{3\cdot 2\sqrt{2}}\right) - \left(-\frac{a^{3/2}}{\sqrt{2}} - \frac{-2a^{3/2}}{3\cdot 2\sqrt{2}}\right) \\
&= \left(\frac{a^{3/2}}{\sqrt{2}} - \frac{1}{3}\cdot\frac{a^{3/2}}{\sqrt{2}}\right) + \left(\frac{a^{3/2}}{\sqrt{2}} - \frac{1}{3}\cdot\frac{a^{3/2}}{\sqrt{2}}\right) = \frac{2}{3}\cdot\frac{a^{3/2}}{\sqrt{2}} + \frac{2}{3}\cdot\frac{a^{3/2}}{\sqrt{2}} \\
&= \frac{4}{3}\cdot\frac{a^{3/2}}{\sqrt{2}} = \frac{2\sqrt{2}}{3}\cdot a^{3/2}
\end{aligned}
$$

Since we want this area to be $\frac{64}{3}$, we need to solve $\frac{2\sqrt{2}}{3}\cdot a^{3/2} = \frac{64}{3}$ for a. Here we go:

$$
\begin{aligned}
\frac{2\sqrt{2}}{3}\cdot a^{3/2} = \frac{64}{3} &\implies a^{3/2} = \frac{64}{3}\cdot\frac{3}{2\sqrt{2}} = \frac{64}{2\sqrt{2}} = \frac{32}{\sqrt{2}} = 16\sqrt{2} \\
&\implies a^3 = \left(a^{3/2}\right)^2 = \left(16\sqrt{2}\right)^2 = 256\cdot 2 = 512 = 2^9 \\
&\implies a = \left(a^3\right)^{1/3} = \left(2^9\right)^{1/3} = 2^{9/3} = 2^3 = 8
\end{aligned}
$$

Thus the real number $a > 0$ such that the area of the finite region below the parabola $y = a - x^2$ and above the parabola $y = x^2$ is exactly $\frac{64}{3}$ is $a = 8$. Note that this makes the intersection points of the parabolas occur at $x = \pm\sqrt{a/2} = \pm\sqrt{8/2} = \pm\sqrt{4} = \pm 2$. $\square$

SOLUTION 2. *(Using SageMath.)* See above for the setup and explanations.

```
[1]: var("a")
     solve(a-x^2 == x^2, x)
```

```
[1]: [x == -sqrt(1/2)*sqrt(a), x == sqrt(1/2)*sqrt(a)]
```

```
[2]: solve( integral( a - 2*x^2, x, -sqrt(1/2)*sqrt(a), sqrt(1/2)*sqrt(a) ) == 64/3,
     ↪a )
```

```
[2]: [a == -4*I*sqrt(3) - 4, a == 4*I*sqrt(3) - 4, a == 8]
```

We discard the complex-valued solutions and keep the real solution $a = 8$. ■

NOTE. This sort of thing, where two or more variables are in play, is why the `solve` and `integral` commands demand that one be explicit about which varaible to solve for or use.