Mathematics 1110H – Calculus I: Limits, Derivatives, and Integrals
TRENT UNIVERSITY, Fall 2023

Assignment #3
The Limit Game and a Differential Equation
Due just before midnight on Friday, 6 October.*

The usual $\varepsilon - \delta$ definition of limits,

> DEFINITION. $\lim_{x \to a} f(x) = L$ exactly when for every $\varepsilon > 0$ there is a $\delta > 0$ such that for any x with $|x - a| < \delta$ we are guaranteed to have $|f(x) - L| < \varepsilon$ as well.

is pretty hard to wrap your head around the first time or three for most people. Here is less common version of the definition, equivalent to the standard one, which recasts the confusing logical structure of the standard definition in terms of a game:

> ALTERNATE DEFINITION. The *limit game* for $f(x)$ at $x = a$ with target L is a three-move game played between two players A and B as follows:
>
> 1. A moves first, picking a small number $\varepsilon > 0$.
> 2. B moves second, picking another small number $\delta > 0$.
> 3. A moves third, picking an x that is within δ of a, *i.e.* $a - \delta < x < a + \delta$.
>
> To determine the winner, we evaluate $f(x)$. If it is within ε of the target L, *i.e.* $L - \varepsilon < f(x) < L + \varepsilon$, then player B wins; if not, then player A wins.
>
> With this idea in hand, $\lim_{x \to a} f(x) = L$ means that player B has a winning strategy in the limit game for $f(x)$ at $x = a$ with target L; that is, if B plays it right, B will win no matter what A tries to do. (Within the rules ... :-) Conversely, $\lim_{x \to a} f(x) \neq L$ means that player A is the one with a winning strategy in the limit game for $f(x)$ at $x = a$ with target L.

The game definition of limits isn't really better or worse that the usual $\varepsilon - \delta$ definition, but each is easier for some people to understand, and the exercise in trying it both ways usually helps in understanding what is really going on with limits.

1. Use the alternate definition of limits to verify that $\lim_{x \to 3} (-3x + 4) = -5$. *[2.5]*

Hint: Try using the usual $\varepsilon - \delta$ definition of limits first to work out a winning strategy for player B. Note that player B only gets one move: picking a δ after player A plays an ε.

SOLUTION. Player B's problem is that player A gets both the first move, picking the $\varepsilon > 0$, and the last move, picking an x with $3 - \delta < x < 3 + \delta$. Player B only gets one move, picking the $\delta > 0$. Note that player B wins only if $-5 - \varepsilon < -3x + 4 < -5 + \varepsilon$, otherwise player A wins.

* You should submit your solutions via Blackboard's Assignments module, preferably as a single pdf. If submission via Blackboard fails, please submit your work to your instructor by email or on paper.

Player B thinks:

What should δ be? What does $3-\delta < x < 3+\delta$ mean for x? Well, I need to make $-5-\varepsilon < -3x+4 < -5+\varepsilon$ happen, so what does that mean for x? Let's see if I can isolate x here.

$$\begin{aligned} -5-\varepsilon < -3x+4 < -5+\varepsilon &\iff -9-\varepsilon < -3x < -9+\varepsilon \\ &\iff 3+\frac{\varepsilon}{3} > x > 3-\frac{\varepsilon}{3} \\ &\iff 3-\frac{\varepsilon}{3} < x < 3+\frac{\varepsilon}{3} \end{aligned}$$

So if I can force A to play an x between $3-\frac{\varepsilon}{3}$ and $3+\frac{\varepsilon}{3}$, I'll win because then $-5-\varepsilon < -3x+4 < -5+\varepsilon$. But I can force A to play such an x if I make $\delta = \frac{\varepsilon}{3}$, so that's what I'll do!

Since player B can devise with a strategy that will win no matter what player A does, $\lim_{x\to 3}(-3x+4) = -5$ by the alternate version of the definition of limit. □

2. Use the alternate definition of limits to verify that $\lim_{x\to -1}(2x-3) \neq 0$. *[2.5]*

Hint: You need a winning strategy for player A. What is a suitable ε to play on the first move that will defeat any δ that player B might choose?

Solution. Again, recall that player A gets both the first move, picking the $\varepsilon > 0$, and the last move, picking an x with $-1-\delta < x < -1+\delta$. Player B only gets one move, picking the $\delta > 0$. This time player B wins if $-\varepsilon = -0-\varepsilon < 2x-3 < -0+\varepsilon = \varepsilon$, while player A wins if either $2x-3 = (2x-3)-0 \leq -\varepsilon$ or $2x-3 = (2x-3)-0 \geq \varepsilon$.

Player A thinks:

OK, so to win I need to pick a $\varepsilon > 0$ and later an x between $-1-\delta$ and $-1+\delta$ so that $|2x-3| \geq \varepsilon$. The problem is that B can play any $\delta > 0$ they like in between my picking ε and then x. So I have to be able to pick an x really close to -1 – what if I try plugging in $x=-1$? Then $|2(-1)-3| = |-5| = 5$. Hmm – so any $\varepsilon > 0$ that separates the real limit of 5 from the wrong limit of 0 should work.

Let's try $\varepsilon = 1$. If B now plays some $\delta > 0$, I need to pick an x between $-1-\delta$ and $-1+\delta$ so that $2x-3 \leq 1$ or $2x-3 \geq 1$. If I pick, say, $x = -1-\frac{\delta}{2}$, then $-1-\delta < x < -1+\delta$ – check! – and $2x-3 = 2\left(-1-\frac{\delta}{2}\right)-3 = -5-\delta < -5 < -1 = -\varepsilon$, so I win!

Since player A can find a strategy that will win no matter what player B does, $\lim_{x\to -1}(2x-3) \neq 0$ according to the alternate definition of limits. □

More on the next page!

If you want to get started on this part of the assignment before attending your lab in MATH 1110H, skimming and later referring to as necessary to Sections 4.22.1 and 4.22.2 of Gregory Bard's book *Sage for Undergraduates* is probably going to be useful. Be advised that the format given there for declaring a function has changed since the book was published nine years ago; please consult the *Glossary of commands* by Maya Peters for the current format. Both documents are in the SageMath folder in the Course Content section on Blackboard.

3. Use SageMath to solve for y as a function of x if $\frac{dy}{dx} = -2y$ and it is also required that $y = 1$ when $x = 0$. *[4]*

Hint: This is a job for the `desolve` command.

SOLUTION. Here we are:

```
[1]: y = function('y')(x)    # We declare y to be a function of x so that we
                             # can differentiate it.
     desolve( diff(y,x) == -2*y, y, ics=[0,1])  # diff(y,x) is the derivative
                             # of y with respect to x.  ics=[0,1] specifies
                             # that y should be 1 when x=0.

[1]: e^(-2*x)
```

That is, SageMath tells us that if $\frac{dy}{dx} = -2y$ and $y = 1$ when $x = 0$, then $y = e^{-2x}$. □

4. Verify by hand that the solution you obtained using SageMath in question **3** is indeed a solution to the given differential equation, $\frac{dy}{dx} = -2y$, with the given initial condition, $y = 0$ when $x = 0$. *[1]*

SOLUTION. We first check whether $y = e^{-2x}$ satisfies the differential equation:

$$\frac{dy}{dx} = \frac{d}{dx}e^{-2x} = e^{-2x} \cdot \frac{d}{dx}(-2x) = e^{-2x} \cdot (-2) = -2e^{-2x} = -2y$$

It does! :-)

We next check whether it also satisfies the initial condition $y = 0$ when $x = 0$:

$$y(0) = e^{-2x}\big|_{x=0} = e^{-2\cdot 0} = e^0 = 1$$

It does! :-)

Thus the solution given by SageMath is indeed a solution to the given differential equation with the given initial condition. ■