**Mathematics 1110H – Calculus I: Limits, Derivatives, and Integrals** (Section C)
TRENT UNIVERSITY, Fall 2021

**Solutions to Quiz #6**
*Wednesday, 3 November.*

Do the following problem.

**1.** Find the domain, any and all intercepts, vertical and horizontal asymptotes, intervals of increase and decrease, local maxima and minima, intervals of concavity, and inflection points, of $f(x) = \dfrac{x^3-1}{x^3+1}$, and sketch its graph. *[5]*

SOLUTION. *i. Domain.* Since $f(x)$ is a rational function, it is defined (as well as continuous and differentiable) for all $x$ except those where the denominator is 0. Since $x^3+1=0$ exactly when $x^3=-1$, *i.e.* exactly when $x=-1$, the domain of $f(x)$ is all $x \neq -1$, that is, $(-\infty,-1)\cup(-1,\infty)$.

*ii. Intercepts.* As $f(0) = \dfrac{0^3-1}{0^3+1} = \dfrac{-1}{1} = -1$, $y=f(x)$ has a $y$-intercept at $y=-1$.

Since $f(x) = \dfrac{x^3-1}{x^3+1} = 0$ when $x^3-1=0$, *i.e.* when $x=1$, $y=f(x)$ has an $x$-intercept at $x=1$.

*iii. Vertical asymptotes.* As noted above, $f(x)$ is defined and continuous for all $x \neq -1$, so the only place there might be a vertical asymptote is at $x=-1$. We compute the necessary limits to check:

$$\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-} \frac{x^3-1 \to -2}{x^3+1 \to 0^-} = +\infty$$

$$\lim_{x\to -1^+} f(x) = \lim_{x\to -1^+} \frac{x^3-1 \to -2}{x^3+1 \to 0^+} = -\infty$$

Thus $f(x)$ does have a vertical asymptote at $x=-1$, approaching $+\infty$ from the left and $-\infty$ from the right.

*iv. Horizontal asymptotes.* We compute the limits as $x\to-\infty$ and as $x\to+\infty$ to check for horizontal asymptotes:

$$\lim_{x\to -\infty} f(x) = \lim_{x\to -\infty} \frac{x^3-1}{x^3+1} = \lim_{x\to -\infty} \frac{x^3-1}{x^3+1}\cdot\frac{\frac{1}{x^3}}{\frac{1}{x^3}} = \lim_{x\to -\infty} \frac{1-\frac{1}{x^3} \to 1^+}{1+\frac{1}{x^3} \to 1^-} = 1^+$$

$$\lim_{x\to +\infty} f(x) = \lim_{x\to +\infty} \frac{x^3-1}{x^3+1} = \lim_{x\to +\infty} \frac{x^3-1}{x^3+1}\cdot\frac{\frac{1}{x^3}}{\frac{1}{x^3}} = \lim_{x\to +\infty} \frac{1-\frac{1}{x^3} \to 1^-}{1+\frac{1}{x^3} \to 1^+} = 1^-$$

It follows that $f(x)$ has $y=1$ as a horizontal asymptote in both directions, approaching it from above as $x\to-\infty$ and from below as $x\to+\infty$.

*v. Intervals of increase and decrease, and local maxima and minima.* We first work out $f'(x)$.

$$f'(x) = \frac{d}{dx}\left(\frac{x^3-1}{x^3+1}\right) = \frac{\left[\frac{d}{dx}\left(x^3-1\right)\right]\left(x^3+1\right) - \left(x^3-1\right)\left[\frac{d}{dx}\left(x^3+1\right)\right]}{\left(x^3+1\right)^2}$$

$$= \frac{3x^2\cdot\left(x^3+1\right) - \left(x^3-1\right)\cdot 3x^2}{\left(x^3+1\right)^2} = \frac{3x^5+3x^2-3x^5+3x^2}{\left(x^3+1\right)^2} = \frac{6x^2}{\left(x^3+1\right)^2}$$

Observe that $f'(x)$ is (like $f(x)$ itself) defined (and continuous and differentiable) for all $x \neq -1$.

It is pretty obvious that $f'(x) = 0$ exactly when $x^2 = 0$, which is true exactly when $x = 0$. Since $6 > 0$ and any square must be at least 0, we have that $f'(x) = \dfrac{6x^2}{\left(x^3+1\right)^2} = 6\left(\dfrac{x}{x^3+1}\right)^2 \geq 0$ for all $x$ for which it is defined. Thus $f'(x)$ is positive, and hence $f(x)$ is increasing, for all $x$ for which it is defined except for $x = 0$. This means that the critical point $x = 0$ is neither a local maximum nor a local minimum. We summarize this information in a table:

| $x$ | $(-\infty, -1)$ | $-1$ | $(-1, 0)$ | $0$ | $(0, \infty)$ |
|---|---|---|---|---|---|
| $f'(x)$ | $+$ | undef. | $+$ | $0$ | $+$ |
| $f(x)$ | $\uparrow$ | undef. | $\uparrow$ | crit. pt. | $\uparrow$ |

*vi. Intervals of concavity and points of inflection.* We first work out $f''(x)$.

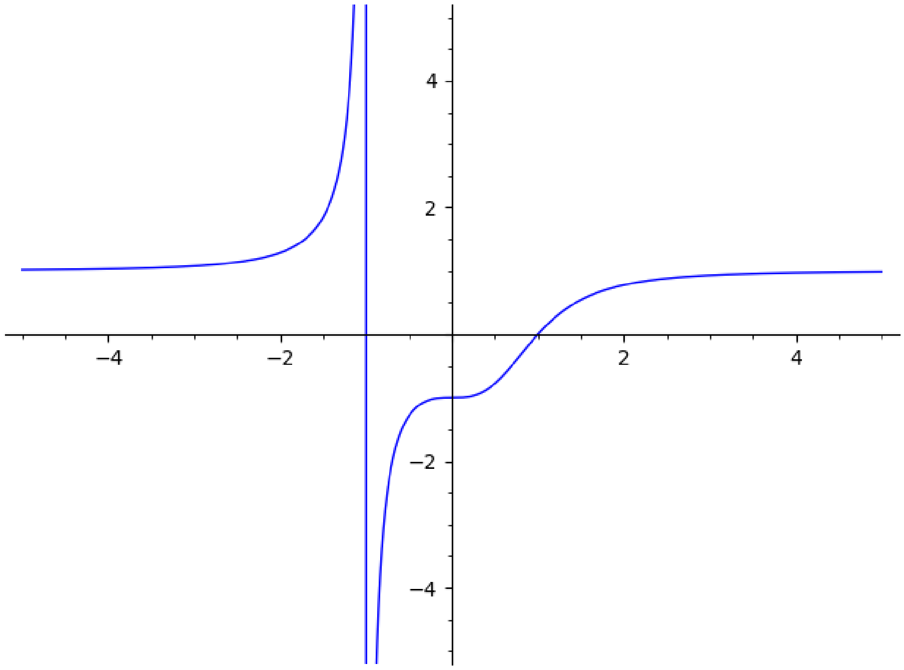
$$\begin{aligned}
f''(x) &= \frac{d}{dx}\left(\frac{6x^2}{\left(x^3+1\right)^2}\right) = \frac{\left[\frac{d}{dx}6x^2\right]\left(x^3+1\right)^2 - 6x^2\left[\frac{d}{dx}\left(x^3+1\right)^2\right]}{\left(\left(x^3+1\right)^2\right)^2} \\
&= \frac{12x\cdot\left(x^3+1\right)^2 - 6x^2\cdot 2\left(x^3+1\right)\cdot\frac{d}{dx}\left(x^3+1\right)}{\left(x^3+1\right)^4} \\
&= \frac{12x\cdot\left(x^3+1\right)^2 - 6x^2\cdot 2\left(x^3+1\right)\cdot 3x^2}{\left(x^3+1\right)^4} = \frac{12x\cdot\left(x^3+1\right) - 6x^2\cdot 2\cdot 3x^2}{\left(x^3+1\right)^3} \\
&= \frac{12x^4 + 12x - 36x^4}{\left(x^3+1\right)^3} = \frac{12x - 24x^4}{\left(x^3+1\right)^3} = \frac{12x\left(1-2x^3\right)}{\left(x^3+1\right)^3}
\end{aligned}$$

Observe that $f'(x)$ (like $f(x)$ and $f'(x)$) is defined (and continuous and differentiable) for all $x \neq -1$.

$f''(x) = 0$ exactly when $12x\left(1-2x^3\right) = 0$, which happens exactly when $x = 0$ or $x = 2^{-1/3} = \dfrac{1}{\sqrt[3]{2}} \approx 0.7937$. When $x < -1$, $12x < 0$, $1 - 2x^3 > 0$ and $\left(x^3+1\right)^3 < 0$, so $f''(x) = \dfrac{12x\left(1-2x^3\right)}{\left(x^3+1\right)^3} > 0$, so $f(x)$ is concave up. Similarly, when $-1 < x < 0$, we have $12x < 0$, $1 - 2x^3 > 0$, and $\left(x^3+1\right)^3 > 0$, so $f''(x) < 0$, and so $f(x)$ is concave down. For $0 < x < 2^{-1/3}$ we have $12x > 0$, $1 - 2x^3 > 0$, and $\left(x^3+1\right)^3 > 0$, so $f''(x) > 0$, and so $f(x)$ is concave up. Finally, when $x > 2^{-1/3}$, we have $12x > 0$, $1 - 2x^3 < 0$, and $\left(x^3+1\right)^3 > 0$, so $f''(x) < 0$ and $f(x)$ is concave down. It follows that both $x = 0$ and $x = 2^{-1/3}$ are inflection points because $f(x)$ switches concavity at each. ($x = -1$ would be an inflection point, too, if $f(x)$ and its derivatives were actually defined there ... ) We summarize this information in another table:

| $x$ | $(-\infty, -1)$ | $-1$ | $(-1, 0)$ | $0$ | $\left(0, 2^{-1/3}\right)$ | $2^{-1/3}$ | $\left(2^{-1/3}, +\infty\right)$ |
|---|---|---|---|---|---|---|---|
| $f''(x)$ | $+$ | undef. | $-$ | $0$ | $+$ | $0$ | $-$ |
| $f(x)$ | $\smile$ | undef. | $\frown$ | infl. pt. | $\smile$ | infl. pt. | $\frown$ |

*vii. The graph.* It's a bit of a cheat, but the following graph was made by `SageMath` using the command: `plot((x^3-1)/(x^3+1),-5,5,ymin=-5,ymax=5)`




Note that `SageMath` drew in the vertical asymptote at $x = -1$ on it's own, even though it isn't part of the graph of the function. This weakness is shared by many other plotting programs. ■

*[Total = 5]*