Mathematics 1110H – Calculus I: Limits, Derivatives, and Integrals (Section C)
Trent University, Fall 2021

Solutions to Quiz #~~3~~ 4 *(with corrections)*
Wednesday, 13 October.

Do all three of the following problems.

1. Find the domain and all of the vertical and horizontal asymptotes, if any, of

$$f(x) = \frac{x^2+3x}{x^2+x-2} \,. \quad [2]$$

Solution. The domain of $f(x)$ consists of all values of x for which the function is defined. In this case, the only way the definition of the function can fail is the denominator is 0, *i.e.* if $x^2+x-2=0$. Since $x^2+x-2=(x+2)(x-1)$, it will equal 0 only when $x=-2$ or $x=1$. Thus the domain of $f(x)$ is $\{\, x \in \mathbb{R} \mid x \neq -2 \text{ and } x \neq 1 \,\} = (-\infty,-2) \cup (-2,1) \cup (1,\infty)$.

Since $f(x)$, like the vast majority of functions we encounter, is continuous wherever it is defined, the only places there might be a vertical asymptote would be at the points where the function is not defined, that is, at $x=-2$ and $x=1$. We take the limit of $f(x)$ from each side at both of these points to check:

$$\lim_{x\to-2^-} \frac{x^2+3x}{x^2+x-2} = \lim_{x\to-2^-} \frac{x(x+3)}{(x-1)(x+2)} \begin{matrix} \to -2 \\ \to 0^+ \end{matrix} = -\infty$$
$$\lim_{x\to-2^+} \frac{x^2+3x}{x^2+x-2} = \lim_{x\to-2^+} \frac{x(x+3)}{(x-1)(x+2)} \begin{matrix} \to -2 \\ \to 0^- \end{matrix} = +\infty$$
$$\lim_{x\to1^-} \frac{x^2+3x}{x^2+x-2} = \lim_{x\to1^-} \frac{x(x+3)}{(x-1)(x+2)} \begin{matrix} \to 4 \\ \to 0^- \end{matrix} = -\infty$$
$$\lim_{x\to1^+} \frac{x^2+3x}{x^2+x-2} = \lim_{x\to1^+} \frac{x(x+3)}{(x-1)(x+2)} \begin{matrix} \to 4 \\ \to 0^+ \end{matrix} = +\infty$$

Thus $f(x)$ has vertical asymptotes at $x=1$ and $x=-2$, aproaching $-\infty$ from the left and $+\infty$ from the right at each point.

Finally, we check the limits of $f(x)$ as $x\to-\infty$ and $x\to\infty$ to see if $f(x)$ has any horizontal asymptotes:

$$\lim_{x\to-\infty} \frac{x^2+3x}{x^2+x-2} = \lim_{x\to-\infty} \frac{x(x+3)}{(x-1)(x+3)} = \lim_{x\to-\infty} \frac{x}{x-1} \cdot \frac{\frac{1}{x}}{\frac{1}{x}} = \lim_{x\to-\infty} \frac{1}{1-\frac{1}{x}} = \frac{1}{1-0} = 1$$
$$\lim_{x\to\infty} \frac{x^2+3x}{x^2+x-2} = \lim_{x\to\infty} \frac{x(x+3)}{(x-1)(x+3)} = \lim_{x\to\infty} \frac{x}{x-1} \cdot \frac{\frac{1}{x}}{\frac{1}{x}} = \lim_{x\to\infty} \frac{1}{1-\frac{1}{x}} = \frac{1}{1-0} = 1$$

Thus $f(x)$ has $y=1$ as a horizontal asymptote in both directions. ■

2. Find the domain and all of the vertical and horizontal asymptotes, if any, of

$$g(x) = e^{-x}\sin(x)\,. \quad \textit{[1.5]}$$

SOLUTION. $\sin(x)$ and e^{-x} are both defined for all $x \in \mathbb{R}$, and hence so is their product, $g(x)$. Thus the domain of $g(x)$ is $\mathbb{R} = (-\infty, \infty)$.

Since $g(x)$ is defined and continuous for all x, it has no vertical asymptotes.

Finally, we check the limits of $g(x)$ as $x \to -\infty$ and $x \to \infty$ to see if $g(x)$ has any horizontal asymptotes:

First, note that as $x \to -\infty$, $\sin(x)$ oscillates between -1 and 1 while $e^{-x} \to \infty$. It follows that $g(x) = e^{-x}\sin(x)$ has increasingly large oscillations as $x \to -\infty$, so $\lim_{x\to-\infty} g(x)$ does not exists. Thus $g(x)$ has no horizontal asymptote as $x \to -\infty$.

Second, note that as $x \to \infty$, $\sin(x)$ oscillates between -1 and 1 while $e^{-x} \to 0^+$. It follows that as $x \to +\infty$, $-e^{-x} \le e^{-x}\sin(x) \le e^{-x}$, and since $\lim_{x\to\infty} e^{-x} = 0$, it follows by the Squeeze Theorem that $\lim_{x\to\infty} g(x) = \lim_{x\to\infty} e^{-x}\sin(x) = 0$. Thus $g(x)$ has $y = 0$ as a horizontal asymptote as $x \to \infty$. ■

3. Find the domain and all of the vertical and horizontal asymptotes, if any, of

$$h(x) = e^{-1/x^2}\,. \quad \textit{[1.5]}$$

SOLUTION. Since e^t is defined for all $t \in \mathbb{R}$ and $-1/x^2$ is defined for all $x \neq 0$, it follows that $h(x) = e^{-1/x^2}$ is defined for all $x \neq 0$. Thus the domain of $h(x)$ is $\{\, x \in \mathbb{R} \mid x \neq 0 \,\} = (-\infty, 0) \cup (0, \infty)$.

Since $h(x)$ is defined and continuous for all points $x \neq 0$, the only place it might have a vertical asymptote is at $x = 0$. We take the limits from each direction at $x = 0$ to check. Note that $-1/x^2 \to -\infty$ both as $x \to 0^-$ and as $x \to 0^+$.

$$\lim_{x\to0^-} h(x) = \lim_{x\to0^-} e^{-1/x^2} = \lim_{t\to-\infty} e^t = 0^+$$
$$\lim_{x\to0^+} h(x) = \lim_{x\to0^+} e^{-1/x^2} = \lim_{t\to-\infty} e^t = 0^+$$

It follows that $h(x)$ does not have a vertical asymptote at $x = 0$. (It has a "removable discontinuity": if we defined $h(0)$ to be 0, the function would be continuous at $x = 0$.) Thus $h(x)$ has no vertical asymptotes.

Finally, we check the limits of $h(x)$ as $x \to -\infty$ and $x \to \infty$ to see if $h(x)$ has any horizontal asymptotes. Note that $-1/x^2 \to 0$ both as $x \to -\infty$ and as $x \to \infty$.

$$\lim_{x\to-\infty} h(x) = \lim_{x\to-\infty} e^{-1/x^2} = \lim_{u\to0^-} e^u = e^{0^-} = 1^-$$
$$\lim_{x\to+\infty} h(x) = \lim_{x\to+\infty} e^{-1/x^2} = \lim_{u\to0^-} e^u = e^{0^-} = 1^-$$

It follows that $h(x)$ has $y = 1$ as a horizontal asymptote, which it approaches from below both as $x \to -\infty$ and as $x \to +\infty$. ■

[Total = 5]