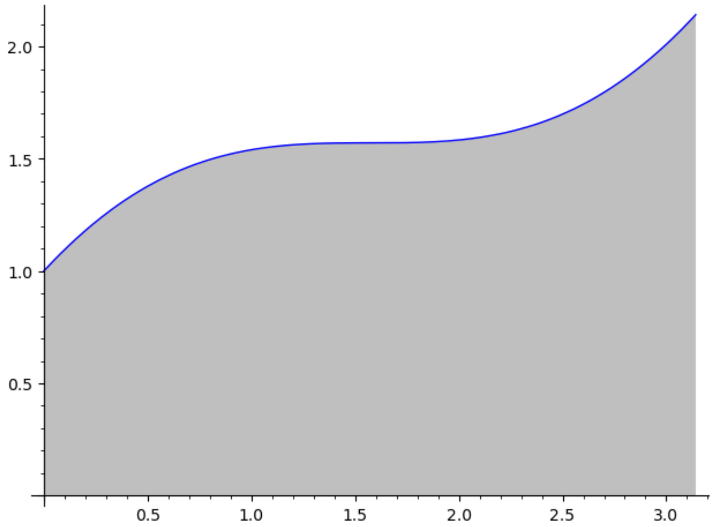
Mathematics 1110H – Calculus I: Limits, Derivatives, and Integrals (Section C)
Trent University, Fall 2021

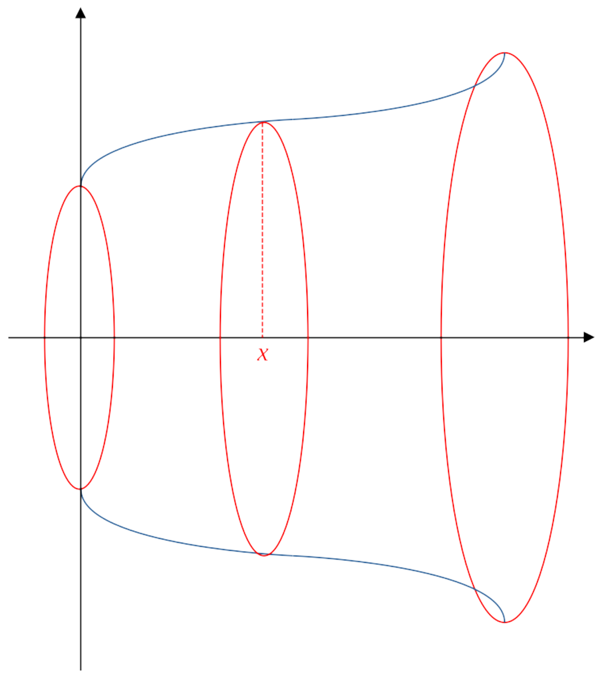
Solution to Quiz #11
Wednesday, 8 December.

1. Consider the region between the curve $y = x + \cos(x)$ and x-axis, where $0 \le x \le \pi$. Find the volume of the solid obtained by revolving this region about the x-axis. *[5]*

A tiny bit of help. Typing `plot(x+cos(x),0,pi,fill=True)` into `SageMath` shows what the region to be revolved about the x-axis looks like:



Solution. Here is a crude sketch of the solid obtained by revolving the given region about the x-axis:



We will use the disk/washer method to find the volume of this solid. Since we are revolving the region about the x-axis, the disk/washer cross-sections are stacked along the x-axis and are perpendicular to it, so we use x as the variable. Recall that $0 \le x \le \pi$ for the given region. The cross-section at x is a disk with radius $r = x+\cos(x)-0 = x+\cos(x)$, so it has area $A(x) = \pi r^2 = \pi\left(x+\cos(x)\right)^2$. Thus the volume of the solid is given by:

$$\begin{aligned} V &= \int_0^\pi A(x)\,dx = \int_0^\pi \pi\left(x+\cos(x)\right)^2\,dx = \pi\int_0^\pi \left(x^2+2x\cos(x)+\cos^2(x)\right)\,dx \\ &= \pi\int_0^\pi x^2\,dx + \pi\int_0^\pi 2x\cos(x)\,dx + \pi\int_0^\pi \cos^2(x)\,dx \end{aligned}$$

For the first of the three integrals, we will use the Power Rule; for the second, we will use integration by parts with $u = 2x$ and $v' = \cos(x)$, so $u' = 2$ and $v = \sin(x)$; and for the third, we will use the trigonometric integral reduction formula $\int \cos^n(x)\,dx = \frac{1}{n}\cos^{n-1}(x)\sin(x) + \frac{n-1}{n}\int \cos^{n-2}(x)\,dx$. Then the volume of the solid is:

$$\begin{aligned} V &= \pi\int_0^\pi x^2\,dx + \pi\int_0^\pi 2x\cos(x)\,dx + \pi\int_0^\pi \cos^2(x)\,dx \\ &= \left.\frac{\pi x^3}{3}\right|_0^\pi + \pi\left[\left.2x\sin(x)\right|_0^\pi - \int_0^\pi 2\sin(x)\,dx\right] + \pi\left[\left.\frac{1}{2}\cos(x)\sin(x)\right|_0^\pi + \frac{1}{2}\int_0^\pi \cos^0(x)\,dx\right] \\ &= \frac{\pi\cdot\pi^3}{3} - \frac{\pi\cdot 0^3}{3} + \pi\left[2\pi\sin(\pi) - 2\cdot 0\sin(0) - \left.\left(-2\cos(x)\right)\right|_0^\pi\right] \\ &\quad + \pi\left[\frac{1}{2}\cos(\pi)\sin(\pi) - \frac{1}{2}\cos(0)\sin(0) + \frac{1}{2}\int_0^\pi 1\,dx\right] \\ &= \frac{\pi^4}{3} - 0 + \pi\left[2\pi\cdot 0 - 0 + \left.2\cos(x)\right|_0^\pi\right] + \pi\left[\frac{1}{2}\cdot(-1)\cdot 0 - \frac{1}{2}\cdot 1\cdot 0 + \left.\frac{x}{2}\right|_0^\pi\right] \\ &= \frac{\pi^4}{3} + \pi\left[2\cos(\pi) - 2\cos(0)\right] + \pi\left[\frac{\pi}{2} - \frac{0}{2}\right] \\ &= \frac{\pi^4}{3} + \pi\left[2\cdot(-1) - 2\cdot 1\right] + \frac{\pi^2}{2} = \frac{\pi^4}{3} - 4\pi + \frac{\pi^2}{2} = \frac{\pi^4}{3} + \frac{\pi^2}{2} - 4\pi \approx 24.8381 \qquad \blacksquare \end{aligned}$$

CHECK. We plug the volume integral into `SageMath` to evaluate it. Typing

```
N(pi*integral((x+cos(x))^2,x,0,pi))
```

into `SageMath` yields

```
24.8381285975196
```

so we seem to have gotten it right.