Mathematics 1110H – Calculus I: Limits, Derivatives, and Integrals
TRENT UNIVERSITY, Fall 2021
Take-Home Final Examination

Available on Blackboard from 12:00 a.m. on Monday, 13 December.
Due on Blackboard by 11:59 p.m. on Wednesday, 15 December.

Submission: Scans or photos of handwritten work are entirely acceptable so long as they are legible and in some common format; solutions submitted as a single pdf are strongly preferred. If submission via Blackboard's Assignments module fails repeatedly, then, as a last resort, email them to the instructor at: `sbilaniuk@trentu.ca`

Allowed aids: For this exam, you are permitted to use your textbook and all other course material, including that on Blackboard and the archive page, from this and any other mathematics course(s) you have taken or are taking now, but *you may not use any other sources or aids, nor give or receive any help*, except to ask the instructor to clarify questions and to use a calculator (any that you like).

Instructions: Do parts **A** and **B**, and, if you wish, part **C**. Please show all your work and justify all your answers. *If in doubt about something,* **ask!**

Part A. Do *all four* (4) of **1**–**4**. *[Subtotal = 72]*

1. Compute $\dfrac{dy}{dx}$ as best you can in any *five* (5) of **a**–**f**. *[20 = 5 × 4 each]*

a. $\cos(x+y)=0$ **b.** $y=\left(x^2+1\right)^{13}$ **c.** $y=\displaystyle\int_{-\sin(x)}^{0}\arcsin(t)\,dt$

d. $y=e^{x(x-1)}$ **e.** $y=\dfrac{x+1}{x^2-1}$ **f.** $y=\left(x^2+1\right)\arctan(x)$

SOLUTIONS. **a.** *A little trigonometry before differentiating.* $\cos(x+y)=0$ exactly when $x+y=\dfrac{\pi}{2}+n\pi$ for some integer n, so $y=-x+k\pi$ and $\dfrac{dy}{dx}=\dfrac{d}{dx}(-x+k\pi)=-1$. □

a. *Implicit differentiation.* $\dfrac{d}{dx}\cos(x+y)=-\sin(x+y)\dfrac{d}{dx}(x+y)=-\sin(x+y)\left(1+\dfrac{dy}{dx}\right)$ and $\dfrac{d}{dx}0=0$, so $-\sin(x+y)\left(1+\dfrac{dy}{dx}\right)=0$. It follows that either $\sin(x+y)=0$ or $1+\dfrac{dy}{dx}=0$. In the latter case, we can solve the equation to get $\dfrac{dy}{dx}=-1$, while in the former case we must have $x+y=k\pi$ for some integer k, so $y=-x+k\pi$ and $\dfrac{dy}{dx}=\dfrac{d}{dx}(-x+k\pi)=-1$. Either way, $\dfrac{dy}{dx}=-1$. □

b. *Power Rule & Chain Rule.* Here we go:

$$\frac{dy}{dx}=\frac{d}{dx}\left(x^2+1\right)^{13}=13\left(x^2+1\right)^{12}\cdot\frac{d}{dx}\left(x^2+1\right)=13\left(x^2+1\right)^{12}\cdot 2x=26x\left(x^2+1\right)^{12} \quad \square$$

c. *The Fundamental Theorem of Calculus and the Chain Rule.* Recall that one version of the Fundamental Theorem of Calculus tells us that $\frac{d}{dx}\int_c^x f(t)\,dt=f(x)$. We will also use the Chain Rule and the facts that $\int_a^b f(t)\,dt = (-1)\int_b^a f(t)\,dt$ and that arcsin is an odd function, *i.e.* $\arcsin(-t) = (-1)\arcsin(t)$, and is the inverse function to $\sin(x)$ for values of x near 0.

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\int_{-\sin(x)}^{0} \arcsin(t)\,dt = \frac{d}{dx}(-1)\int_{0}^{-\sin(x)} \arcsin(t)\,dt \\ &= (-1)\arcsin\left(-\sin(x)\right)\cdot\frac{d}{dx}\left(-\sin(x)\right) \\ &= (-1)(-1)\arcsin(\sin(x))\cdot(-\cos(x)) = -x\cos(x) \quad \Box\end{aligned}$$

d. *Chain Rule and Power Rule.* Here we go:

$$\frac{dy}{dx} = \frac{d}{dx}e^{x(x-1)} = e^{x^2-x}\cdot\frac{d}{dx}x(x-1) = e^{x(x-1)}\cdot\frac{d}{dx}\left(x^2-x\right) = e^{x(x-1)}(2x-1) \quad \Box$$

e. *Quotient Rule.* Here we go:

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\left(\frac{x+1}{x^2-1}\right) = \frac{\left[\frac{d}{dx}(x+1)\right]\left(x^2-1\right)-(x+1)\left[\frac{d}{dx}\left(x^2-1\right)\right]}{\left(x^2-1\right)^2} \\ &= \frac{1\cdot\left(x^2-1\right)-(x+1)\cdot 2x}{\left(x^2-1\right)^2} = \frac{x^2-1-\left(2x^2-2x\right)}{\left(x^2-1\right)^2} = \frac{-x^2-2x-1}{\left(x^2-1\right)^2} \\ &= \frac{-(x+1)^2}{\left(x^2-1\right)^2} = -\left(\frac{x+1}{x^2-1}\right)^2 = -\left(\frac{x+1}{(x-1)(x+1)}\right)^2 = -\left(\frac{1}{x-1}\right)^2 \\ &= \frac{-1}{(x-1)^2} \quad \Box\end{aligned}$$

e. *Simplification, Power Rule, and Chain Rule.* Here we go:

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\left(\frac{x+1}{x^2-1}\right) = \frac{d}{dx}\left(\frac{x+1}{(x-1)(x+1)}\right) = \frac{d}{dx}\left(\frac{1}{x-1}\right) = \frac{d}{dx}(x-1)^{-1} \\ &= (-1)(x-1)^{-2}\cdot\frac{d}{dx}(x-1) = -(x-1)^{-2}\cdot 1 = \frac{-1}{(x-1)^2} \quad \Box\end{aligned}$$

f. *Product Rule and Power Rule.* Here we go:

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\left(x^2+1\right)\arctan(x) = \left[\frac{d}{dx}\left(x^2+1\right)\right]\cdot\arctan(x) + \left(x^2+1\right)\cdot\left[\frac{d}{dx}\arctan(x)\right] \\ &= 2x\cdot\arctan(x) + \left(x^2+1\right)\cdot\frac{1}{x^2+1} = 2x\arctan(x)+1 \quad \blacksquare\end{aligned}$$

2. Evaluate any *five* (5) of the integrals **a**–**f**. *[20 = 5 × 4 each]*

a. $\displaystyle\int_1^e \ln\left(x^{17}\right)\,dx$ **b.** $\displaystyle\int_0^{\pi/8} \sec^3(2x)\,dx$ **c.** $\displaystyle\int_{-1}^1 \frac{x^2-1}{x^4-1}\,dx$

d. $\displaystyle\int \frac{(\ln(x)+1)^2}{2x}\,dx$ **e.** $\displaystyle\int x\sec^2 x\,dx$ **f.** $\displaystyle\int x^x\cdot(\ln(x)+1)\,dx$

SOLUTIONS. **a.** *Integration by parts.* After a small application of the properties of logarithms, we will integration by parts with $u = \ln(x)$ and $v' = 17$, so $u' = \frac{1}{x}$ and $v = 17x$.

$$\begin{aligned}
\int_1^e \ln\left(x^{17}\right)\,dx &= \int_1^e 17\ln(x)\,dx = \ln(x)\cdot 17x\Big|_1^e - \int_1^e \frac{1}{x}\cdot 17x\,dx \\
&= 17\cdot e\cdot\ln(e) - 17\cdot 1\cdot\ln(1) - \int_1^e 17\,dx \\
&= 17e\cdot 1 - 17\cdot 0 - 17x\Big|_1^e = 17e - 0 - (17\cdot e - 17\cdot 1) \\
&= 17e - 17e + 17 = 17 \quad \square
\end{aligned}$$

b. *Substitution and a trigonometric integral reduction formula.* We will use the subsitution $w = 2x$, so $dw = 2\,dx$ and $dx = \frac{1}{2}\,dw$, changing the limits as we go along: $\begin{array}{ccc} x & 0 & \pi/8 \\ w & 0 & \pi/4 \end{array}$ After this, we will apply the integral reduction formula for powers of sec.

$$\begin{aligned}
\int_0^{\pi/8} \sec^3(2x)\,dx &= \int_0^{\pi/4} \sec^3(w)\,\frac{1}{2}\,dw = \frac{1}{2}\int_0^{\pi/4} \sec^3(w)\,dw \\
&= \frac{1}{2}\left[\frac{1}{3-1}\tan(w)\sec^{3-2}(w)\Big|_0^{\pi/4} + \frac{3-2}{3-1}\int_0^{\pi/4} \sec^{3-2}(w)\,dw\right] \\
&= \frac{1}{2}\left[\frac{1}{2}\tan(w)\sec(w)\Big|_0^{\pi/4} + \frac{1}{2}\int_0^{\pi/4} \sec(w)\,dw\right] \\
&= \frac{1}{4}\tan(w)\sec(w)\Big|_0^{\pi/4} + \frac{1}{4}\ln\left(\sec(w)+\tan(w)\right)\Big|_0^{\pi/4} \\
&= \frac{1}{4}\tan\left(\frac{\pi}{4}\right)\sec\left(\frac{\pi}{4}\right) - \frac{1}{4}\tan(0)\sec(0) \\
&\qquad + \frac{1}{4}\ln\left(\tan\left(\frac{\pi}{4}\right)+\sec\left(\frac{\pi}{4}\right)\right) - \frac{1}{4}\ln\left(\tan(0)+\sec(0)\right) \\
&= \frac{1}{4}\cdot 1\cdot\sqrt{2} - \frac{1}{4}\cdot 0\cdot 1 + \frac{1}{4}\ln\left(1+\sqrt{2}\right) - \frac{1}{4}\ln(0+1) \\
&= \frac{\sqrt{2}}{4} - 0 + \frac{1}{4}\ln\left(1+\sqrt{2}\right) - \frac{1}{4}\ln(1) \\
&= \frac{\sqrt{2}}{4} + \frac{1}{4}\ln\left(1+\sqrt{2}\right) - 0 = \frac{\sqrt{2}+\ln\left(1+\sqrt{2}\right)}{4} \quad \square
\end{aligned}$$

c. *Simplify the integrand.* Here we go:

$$\begin{aligned}\int_{-1}^{1} \frac{x^2-1}{x^4-1}\,dx &= \int_{-1}^{1} \frac{x^2-1}{(x^2-1)(x^2+1)}\,dx = \int_{-1}^{1} \frac{1}{x^2+1}\,dx \\ &= \arctan(x)\big|_{-1}^{1} = \arctan(1) - \arctan(-1) = \frac{\pi}{4} - \left(-\frac{\pi}{4}\right) = \frac{\pi}{2} \quad \square\end{aligned}$$

d. *Substitution and Power Rule.* We will use the substitution $u = \ln(x)+1$, so $du = \frac{1}{x}\,dx$.

$$\int \frac{(\ln(x)+1)^2}{2x}\,dx = \int \frac{u^2}{2}\,du = \frac{1}{2}\cdot\frac{u^3}{3} + C = \frac{u^3}{6} + C = \frac{(\ln(x)+1)^3}{6} + C \quad \square$$

e. *Integration by parts and Substitution.* We will use integration by parts with $u = x$ and $v' = \sec^2(x)$, so $u' = 1$ and $v = \tan(x)$. Afterwards, we will use the substitution $w = \cos(x)$, so $dw = -\sin(x)\,dx$ and $\sin(x)\,dx = (-1)\,dw$.

$$\begin{aligned}\int x\sec^2 x\,dx &= x\tan(x) - \int 1\cdot\tan(x)\,dx = x\tan(x) - \int \frac{\sin(x)}{\cos(x)}\,dx \\ &= x\tan(x) - \int \frac{1}{w}(-1)\,dw = x\tan(x) + \int \frac{1}{w}\,dw \\ &= x\tan(x) + \ln(w) + C = x\tan(x) + \ln(\cos(x)) + C \quad \square\end{aligned}$$

f. *A little algebra and Substitution.* We will use the fact that $x^x = \left(e^{\ln(x)}\right)^x = e^{x\ln(x)}$, and then substitute $w = x\ln(x)$, so $dw = \left(\frac{d}{dx}x\ln(x)\right)\,dx = \left(1\cdot\ln(x) + x\cdot\frac{1}{x}\right)\,dx = (\ln(x)+1)\,dx$.

$$\begin{aligned}\int x^x\cdot(\ln(x)+1)\,dx &= \int e^{x\ln(x)}\cdot(\ln(x)+1)\,dx = \int e^w\,dw \\ &= e^w + C = e^{x\ln(x)} + C = x^x + C \quad \square\end{aligned}$$

3. Do any *five* (5) of **a–i**. *[20 = 5 × 4 each]*

a. Find all the local maxima and minima, if any, of $y = x^4 - 18x^2$.

b. Sketch the region whose border consists of the curves $y = x^2$ for $0 \le x \le 1$, $y = 2 - x$ for $1 \le x \le 2$, and $y = -\sqrt{1-(x-1)^2}$ for $0 \le x \le 2$, and find its area.

c. Use the ε–δ definition of limits to verify that $\lim_{x \to 1} \sqrt{|x-1|} = 0$.

d. Sketch the solid obtained by revolving the region between $y = \ln(x)$ and $y = 0$, for $1 \le x \le e$, about the x-axis, and find the volume of this solid.

e. Find any and all vertical and horizontal asymptotes of $y = \dfrac{1}{x+1} + \dfrac{1}{x-1}$.

f. Compute $\lim_{x \to \infty} x^3 e^{-x}$.

g. Use the limit definition of the derivative to show that $\dfrac{d}{dx}\left(\dfrac{1}{x^2}\right) = -\dfrac{2}{x^3}$.

h. A rectangular box is 1 m wide, x m long, and y m high. What is the minimum possible surface area of such a box if has a volume of 4 m^3?

i. Show that $\ln(\sec(x) - \tan(x)) = -\ln(\sec(x) + \tan(x))$.

SOLUTIONS. **a.** Note that $y = x^4 - 18x^2$ is a ploynomial, so it is defined, as well as continuous and differentiable, for all x. To find all of its local maxima and minima, we need to find all of its critical points and intervals of increase and decrease. This will require the first derivative:

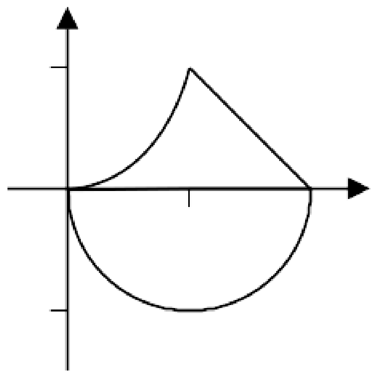
$$\frac{dy}{dx} = \frac{d}{dx}\left(x^4 - 18x^2\right) = 4x^3 - 36x = 4x\left(x^2 - 9\right) = 4x(x+3)(x-3)$$

Obviously, $\dfrac{dy}{dx} = 4x(x+3)(x-3) = 0$ exactly when $x = 0$ or $x = \pm 3$. To see whether these critical points are local maxima or minima, or perhaps neither, we check to see whether $\dfrac{dy}{dx}$ is positive or negative to either side of each critical point:

- For $x < -3$ we have $x < 0$, $x + 3 < 0$, and $x - 3 < 0$, so $\dfrac{dy}{dx} = 4x(x+3)(x-3) < 0$, and so the graph is decreasing.
- For $-3 < x < 0$, we have $x < 0$, $x+3 > 0$, and $x-3 < 0$, so $\dfrac{dy}{dx} = 4x(x+3)(x-3) > 0$, and so the graph is increasing.
- For $0 < x < 3$ we have $x > 0$, $x + 3 > 0$, and $x - 3 < 0$, so $\dfrac{dy}{dx} = 4x(x+3)(x-3) < 0$, and so the graph is decreasing.
- For $x > 3$, we have $x > 0$, $x + 3 > 0$, and $x - 3 > 0$, so $\dfrac{dy}{dx} = 4x(x+3)(x-3) > 0$, and so the graph is increasing.

It follows that $x = -3$ is a local minimum, $x = 0$ is a local maximum, and $x = 3$ is another local minimum of $y = x^4 - 18x^2$. □

b. Here is a sketch of the region:



The area of the region can be most easily computed by breaking the region down into three parts:

- the subregion between $y = x^2$ and the x-axis for $0 \leq x \leq 1$,
- the subregion between $y = 2 - x$ and the x-axis for $1 \leq x \leq 2$, which is a triangle with height and base 1, and
- the subregion between $y = -\sqrt{1-(x-1)^2}$ and the x-axis for $0 \leq x \leq 2$, which is the lower half of the unit circle centred at $(1, 0)$.

The last two subregions have easily computed areas: the area of the triangle is $\frac{1}{2} \cdot \text{base} \cdot \text{height} = \frac{1}{2} \cdot 1 \cdot 1 = \frac{1}{2}$, and the area of half of a unit circle is $\frac{1}{2} \cdot \pi r^2 = \frac{1}{2} \cdot \pi 1^2 = \frac{\pi}{2}$. To compute the area of the first subregion, we – finally! :-) – resort to calculus and compute the area integral:

$$\int_0^1 x^2\,dx = \left.\frac{x^3}{3}\right|_0^1 = \frac{1^3}{3} - \frac{0^3}{3} = \frac{1}{3}$$

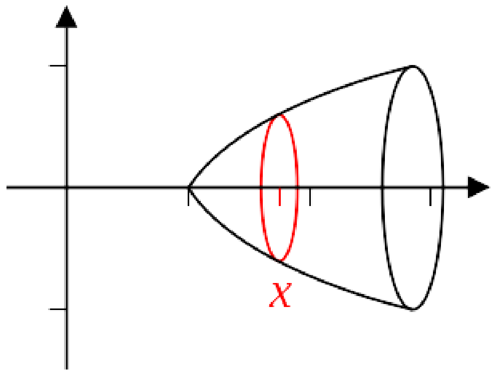
Thus the area of the given region is $\frac{1}{3} + \frac{1}{2} + \frac{\pi}{2} = \frac{5}{6} + \frac{\pi}{2}$. $\square$

c. To verify that $\lim_{x\to 1} \sqrt{|x-1|} = 0$ using the ε–δ definition of limits, we need to show that given any $\varepsilon > 0$, there is a $\delta > 0$ such that if $|x-1| < \delta$, then $\left|\sqrt{|x-1|} - 0\right| < \varepsilon$. We will attempt to work backward from the last condition to determine what δ should be in terms of ε. Suppose that $\varepsilon > 0$ is given. Then:

$$\begin{aligned}
& \left|\sqrt{|x-1|} - 0\right| < \varepsilon \\
\iff & \left|\sqrt{|x-1|}\right| < \varepsilon \\
\iff & \sqrt{|x-1|} < \varepsilon \qquad \text{(Since we're taking the positive root.)} \\
\iff & |x-1| < \varepsilon^2 \qquad \text{(Since } x^2 \text{ is an increasing function for } x > 0.\text{)}
\end{aligned}$$

Since every step above is reversible, if we let $\delta = \varepsilon^2$, it will follow that if $|x-1| < \delta = \varepsilon^2$, then $\left|\sqrt{|x-1|} - 0\right| < \varepsilon$, as required. Thus, by the ε–δ definition of limits, $\lim_{x\to 1} \sqrt{|x-1|} = 0$. $\square$

d. Here is a sketch of the solid, with a cross-section at x drawn in for good measure:



We will use the disk/washer method to fint the volume of this solid. Since we revolved the given region about the x-axis, the disk cross-sections are perpendicular to the axis, and so we use x as our variable. Note that the disk at x has radius $r = y = \ln(x)$ and hence has area $A(x) = \pi r^2 = \pi \ln^2(x)$. This gives us the volume integral $V = \int_1^e A(x)\,dx = \int_1^e \pi \ln^2(x)\,dx = \pi \int_1^e \ln^2(x)\,dx$, which we will compute in two stages. In the first stage, we will use integration by parts with $u = \ln^2(x)$ and $v' = 1$, so $u' = 2\ln(x) \cdot \frac{1}{x}$ and $v = x$. This gives:

$$\begin{aligned} V &= \pi \int_1^e \ln^2(x)\,dx = \pi \left[\ln^2(x) \cdot x\Big|_1^e - \int_1^e 2\ln(x) \cdot \frac{1}{x} \cdot x\,dx \right] \\ &= \pi \left[e\ln^2(e) - 1\ln^2(1) - \int_1^e 2\ln(x)\,dx \right] = \pi \left[e \cdot 1^2 - 1 \cdot 0^2 - 2\int_1^e \ln(x)\,dx \right] \\ &= \pi \left[e - 2\int_1^e \ln(x)\,dx \right] = \pi e - 2\pi \int_1^e \ln(x)\,dx \end{aligned}$$

We will use parts again to deal with the remaining integral, with $s = \ln(x)$ and $t' = 1$, so $s' = \frac{1}{x}$ and $t = x$. This, in turn, gives:

$$\begin{aligned} V &= \pi \int_1^e \ln^2(x)\,dx = \pi e - 2\pi \int_1^e \ln(x)\,dx = \pi e - 2\pi \left[\ln(x) \cdot x\Big|_1^e - \int_1^e \frac{1}{x} \cdot x\,dx \right] \\ &= \pi e - 2\pi \left[e\ln(e) - 1\ln(1) - \int_1^e 1\,dx \right] = \pi e - 2\pi \left[e \cdot 1 - 1 \cdot 0 - x|_1^e \right] \\ &= \pi e - 2\pi \left[e - 0 - (e-1) \right] = \pi e - 2\pi \left[e - e + 1 \right] = \pi e - 2\pi \cdot 1 = (e-2)\pi \quad \square \end{aligned}$$

e. Observe that $y = \frac{1}{x+1} + \frac{1}{x-1}$ is defined and continuous for all x for which it is defined, that is, for all x except for $x = \pm 1$. We take the limits from each side at the exceptional points to check for vertical asymptotes:

$$\lim_{x\to-1^-}\left(\frac{1}{x+1}+\frac{1}{x-1}\right)=\lim_{x\to-1^-}\frac{1}{x+1}\begin{matrix}\to 1\\ \to 0^-\end{matrix}+\lim_{x\to-1^-}\frac{1}{x-1}\begin{matrix}\to 1\\ \to -2\end{matrix}=-\infty-\frac{1}{2}=-\infty$$
$$\lim_{x\to-1^+}\left(\frac{1}{x+1}+\frac{1}{x-1}\right)=\lim_{x\to-1^+}\frac{1}{x+1}\begin{matrix}\to 1\\ \to 0^+\end{matrix}+\lim_{x\to-1^+}\frac{1}{x-1}\begin{matrix}\to 1\\ \to -2\end{matrix}=+\infty-\frac{1}{2}=+\infty$$
$$\lim_{x\to+1^-}\left(\frac{1}{x+1}+\frac{1}{x-1}\right)=\lim_{x\to+1^-}\frac{1}{x+1}\begin{matrix}\to 1\\ \to 2\end{matrix}+\lim_{x\to+1^-}\frac{1}{x-1}\begin{matrix}\to 1\\ \to 0^-\end{matrix}=\frac{1}{2}-\infty=-\infty$$
$$\lim_{x\to-1^+}\left(\frac{1}{x+1}+\frac{1}{x-1}\right)=\lim_{x\to+1^+}\frac{1}{x+1}\begin{matrix}\to 1\\ \to 2\end{matrix}+\lim_{x\to+1^+}\frac{1}{x-1}\begin{matrix}\to 1\\ \to 0^+\end{matrix}=\frac{1}{2}+\infty=+\infty$$

Thus $y=\dfrac{1}{x+1}+\dfrac{1}{x-1}$ has vertical asymptotes at $x=\pm 1$; at each point it approaches $-\infty$ from the left and $+\infty$ from the right.

To check for horizontal asymptotes we take the limits in each direction:

$$\lim_{x\to-\infty}\left(\frac{1}{x+1}+\frac{1}{x-1}\right)=\lim_{x\to-\infty}\frac{1}{x+1}\begin{matrix}\to 1\\ \to -\infty\end{matrix}+\lim_{x\to-\infty}\frac{1}{x-1}\begin{matrix}\to 1\\ \to -\infty\end{matrix}=0^-+0^-=0^-$$
$$\lim_{x\to+\infty}\left(\frac{1}{x+1}+\frac{1}{x-1}\right)=\lim_{x\to+\infty}\frac{1}{x+1}\begin{matrix}\to 1\\ \to +\infty\end{matrix}+\lim_{x\to+\infty}\frac{1}{x-1}\begin{matrix}\to 1\\ \to +\infty\end{matrix}=0^++0^+=0^+$$

Thus $y=\dfrac{1}{x+1}+\dfrac{1}{x-1}$ has $y=0$ as a horizontal asymptote in both directions, which it approaches from below as $x\to-\infty$ and from above as $x\to+\infty$. $\square$

f. We will apply l'Hôpital's Rule three times:

$$\lim_{x\to\infty}x^3e^{-x}=\lim_{x\to\infty}\frac{x^3}{e^x}\begin{matrix}\to\infty\\ \to\infty\end{matrix}=\lim_{x\to\infty}\frac{\frac{d}{dx}x^3}{\frac{d}{dx}e^x}=\lim_{x\to\infty}\frac{3x^2}{e^x}\begin{matrix}\to\infty\\ \to\infty\end{matrix}=\lim_{x\to\infty}\frac{\frac{d}{dx}3x^2}{\frac{d}{dx}e^x}$$
$$=\lim_{x\to\infty}\frac{6x}{e^x}\begin{matrix}\to\infty\\ \to\infty\end{matrix}=\lim_{x\to\infty}\frac{\frac{d}{dx}6x}{\frac{d}{dx}e^x}=\lim_{x\to\infty}\frac{6}{e^x}\begin{matrix}\to 6\\ \to\infty\end{matrix}=0\quad\square$$

g. We will plug $\dfrac{1}{x^2}$ into the limit definition of the derivative and see what happens:

$$\frac{d}{dx}\left(\frac{1}{x^2}\right)=\lim_{h\to0}\frac{\frac{1}{(x+h)^2}-\frac{1}{x^2}}{h}=\lim_{h\to0}\frac{\frac{x^2-(x+h)^2}{(x+h)^2x^2}}{h}=\lim_{h\to0}\frac{x^2-\left(x^2+2hx+h^2\right)}{h(x+h)^2x^2}$$
$$=\lim_{h\to0}\frac{-2hx-h^2}{h(x+h)^2x^2}=\lim_{h\to0}\frac{-2x-h}{(x+h)^2x^2}=\frac{-2x-0}{(x+0)^2x^2}=\frac{-2x}{x^4}=-\frac{2}{x^3}\quad\square$$

h. The volume of a box that is 1 m wide, x m long, and y m high is $V=1\cdot x\cdot y=xy$, and its surface area is the sum of the areas of its six faces, $A=2\cdot1\cdot x+2\cdot1\cdot y+2\cdot x\cdot y=2x+2y+2xy$. Since we are told that the volume of the box is $V=4\ m^3$, we have $V=xy=4$, so $y=\dfrac{4}{x}$.

As a function of x, the surface area is then $A(x) = 2x + 2 \cdot \frac{4}{x} + 2x \cdot \frac{4}{x} = 2x + \frac{8}{x} + 8$. Note that we must have $x > 0$ for the problem to make sense, but that there are no other restrictions, *i.e.* we have $0 < x < \infty$. We need to find the minimum of $A(x)$ for x in this range. Now

$$A'(x) = \frac{d}{dx}\left(2x + \frac{8}{x} + 8\right) = 2 + \frac{-8}{x^2} + 0 = 2 - \frac{8}{x^2},$$

so

$$A'(x) = 0 \iff 2 - \frac{8}{x^2} = 0 \iff 2x^2 = 8 \iff x^2 = 4 \iff x = \pm 2,$$

and since we must have $x > 0$, the only critical point we need to consider is $x = 2$. Note that when $0 < x < 2$, $\frac{8}{x^2} > \frac{8}{2^2} = \frac{8}{4} = 2$, so $A'(x) = 2 - \frac{8}{x^2} < 0$, and when $x > 0$, $\frac{8}{x^2} < \frac{8}{2^2} = \frac{8}{4} = 2$, so $A'(x) = 2 - \frac{8}{x^2} > 0$. This means that $A(x)$ is decreasing for $0 < x < 2$ and increasing for $x > 2$, so $A(2) = 2 \cdot 2 + \frac{8}{2} + 8 = 4 + 4 + 8 = 16$ is the minimum value of $A(x)$. Thus the minimum possible surface area of a box that is 1 m wide, x m long, and y m high and has volume 4 m^3 is 16 m^2. $\square$

i. This is a cute exercise in algebra, using a trigonometric identity – $\sec^2(x) = 1 + \tan^2(x)$ – and a property of logarithms – $\ln\left(a^b\right) = b\ln(a)$ – along the way:

$$\begin{aligned}\ln\left(\sec(x) - \tan(x)\right) &= \ln\left((\sec(x) - \tan(x)) \cdot \frac{\sec(x) + \tan(x)}{\sec(x) + \tan(x)}\right) = \ln\left(\frac{\sec^2(x) - \tan^2(x)}{\sec(x) + \tan(x)}\right)\\ &= \ln\left(\frac{1}{\sec(x) + \tan(x)}\right) = \ln\left((\sec(x) + \tan(x))^{-1}\right)\\ &= (-1)\ln\left(\sec(x) + \tan(x)\right) = -\ln\left(\sec(x) + \tan(x)\right) \quad \blacksquare\end{aligned}$$

4. Find the domain as well as any (and all) intercepts, vertical and horizontal asymptotes, intervals of increase, decrease and concavity, and maximum, minimum, and inflection points of $f(x) = e^{-1/x}$, and sketch its graph based on this information. *[12]*

SOLUTION. We run through the given checklist:

i. *(Domain)* Since $-1/x$ is defined (and continuous and differentiable) for all $x \neq 0$ and e^t is defined (and continuous and differentiable) for all t, $f(x) = e^{-1/x}$ is defined (and continuous and differentiable) for all $x \neq 0$.

ii. *(Intercepts)* $f(x) = e^{-1/x}$ is not defined at $x = 0$, so there is no y-intercept. Since $e^t > 0$ for all t, $f(x) = e^{-1/x} > 0$ for all $x \neq 0$, so there is no x-intercept either.

iii. *(Vertical asymptotes)* Since $f(x) = e^{-1/x}$ is defined and continuous for all $x \neq 0$, the only place there might be a vertical asymptote is at $x = 0$. Let's check:

$$\lim_{x \to 0^+} e^{-1/x} = 0 \quad \text{since } -\frac{1}{x} \to -\infty \text{ as } x \to 0^+ \text{ and } e^t \to 0 \text{ as } t = -\frac{1}{x} \to -\infty$$

$$\lim_{x \to 0^-} e^{-1/x} = +\infty \quad \text{since } -\frac{1}{x} \to +\infty \text{ as } x \to 0^- \text{ and } e^t \to +\infty \text{ as } t = -\frac{1}{x} \to +\infty$$

It follows that $f(x) = e^{1/x}$ has a vertical asymptote on the negative side of $x = 0$, but no vertical asymptote on the positive side.

iv. *(Horizontal asymptotes)* Let's check:

$$\lim_{x \to +\infty} e^{-1/x} = 1^- \quad \text{since } -\frac{1}{x} \to 0^- \text{ as } x \to +\infty \text{ and } e^t \to 1^- \text{ as } t = -\frac{1}{x} \to 0^-$$
$$\lim_{x \to -\infty} e^{-1/x} = 1^+ \quad \text{since } -\frac{1}{x} \to 0^+ \text{ as } x \to -\infty \text{ and } e^t \to 1 \text{ as } t = -\frac{1}{x} \to 0^+$$

Thus $f(x) = e^{-1/x}$ has a horizontal asymptote of $y = 1$ in both directions, approaching it from below in the positive direction and from above in the negative direction.

v. *(Maxima and minima)* $f'(x) = \dfrac{d}{dx} e^{-1/x} = e^{-1/x} \cdot \dfrac{d}{dx}\left(-\dfrac{1}{x}\right) = e^{-1/x}\left(-\dfrac{-1}{x^2}\right) = \dfrac{e^{-1/x}}{x^2}$ is, like $f(x) = e^{-1/x}$, defined and continuous for all $x \neq 0$. Note that since $e^{-1/x} > 0$ and $x^2 > 0$ for all $x \neq 0$, $f'(x) > 0$ for all $x \neq 0$. It follows that $f(x)$ is increasing for all x for which it is defined; in particular, it has no critical points and no local maxima or minima.

vi. *(Curvature and inflection)* First,

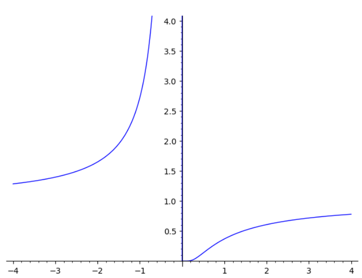
$$f''(x) = \frac{d}{dx}\left(\frac{e^{-1/x}}{x^2}\right) = \frac{\left(\frac{d}{dx} e^{-1/x}\right) x^2 - e^{-1/x}\left(\frac{d}{dx} x^2\right)}{(x^2)^2} = \frac{\frac{e^{-1/x}}{x^2} x^2 - 2x e^{1/x}}{x^4}$$
$$= \frac{e^{-1/x} - 2x e^{-1/x}}{x^4} = \frac{(1-2x)e^{-1/x}}{x^4},$$

which is defined, just as $f(x)$ and $f'(x)$ are, for all $x \neq 0$. Note that since $e^{-1/x} > 0$ whenever it is defined, $f''(x) = 0$ exactly when $1 - 2x = 0$, *i.e.* when $x = \frac{1}{2}$. Since $x^4 > 0$ for all x, we also have that $f''(x) = \frac{(1-2x)e^{1/x}}{x^4} \gtrless 0$ exactly when $1 - 2x \gtrless 0$, *i.e.* exactly when $x \lessgtr \dfrac{1}{2}$. Putting this information in the usual table gives us:

x	$(-\infty, 0)$	0	$\left(0, \frac{1}{2}\right)$	$\frac{1}{2}$	$\left(\frac{1}{2}, +\infty\right)$
$f''(x)$	$-$	undef.	$+$	0	$-$
$f(x)$	$\smile$	undef.	$\smile$	infl. pt.	$\frown$

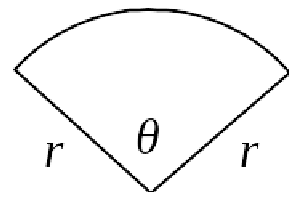
Thus $f(x)$ has one inflection point, at $x = \frac{1}{2}$.

vii. *(Graph)* Using `SageMath`, typing in `plot(e^(-1/x),-4,4,ymin=0,ymax=4)` gets:



$\blacksquare$

Part B. Do any *two* (2) of **5–8**. *[Subtotal = 28 = 2 × 14 each]*



5. A pie slice with angle θ *rad* at the tip is cut out of a circle of radius r. What is the minimum possible perimeter of such a slice if it has an area of 16 cm^2?

SOLUTION. The perimeter of a circle of radius r is its circumference $2\pi r$, and the area of this circle is given by πr^2. The length of the circular arc subtended by the angle θ has the same proportion of the circumference that θ has of the total angle required to go around the entire circle, namely 2π. Thus the length of the circular arc subtended by the angle θ is $2\pi r \cdot \frac{\theta}{2\pi} = \theta r$. It follows that the perimeter of the pie slice is given by $r + r + r\theta = (2+\theta)r$. Similarly, the area of the pie slice has the same proportion of the area of the circle that θ has of the total angle required to go around the entire circle, namely 2π. Thus the area of the pie slice is given by $\pi r^2 \cdot \frac{\theta}{2\pi} = \frac{\theta r^2}{2}$.

If the pie slice has area 16 cm^2, we have $\frac{\theta r^2}{2} = 16$, so $\theta r^2 = 32$, and hence $\theta = \frac{32}{r^2}$. This means that we can write the perimeter of the pie slice as a function of r, $P(r) = (2+\theta)r = \left(2 + \frac{32}{r^2}\right) r = 2r + \frac{32}{r}$. In principle, r could have any value between 0 and ∞, so we need to find the minimum of $P(r)$ for $0 < r < \infty$. Since

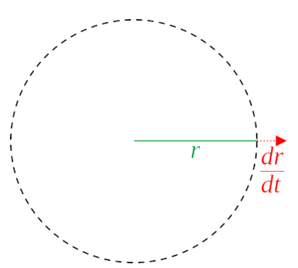
$$P'(r) = \frac{d}{dr}\left(2r + \frac{32}{r}\right) = 2 - \frac{32}{r^2},$$

we have $P'(r) = 0 \iff 2 - \frac{32}{r^2} = 0 \iff r^2 = \frac{32}{2} = 16 \iff r = \pm 4$. Since we are only interested in positive perimeters, we can ignore the critical point $r = -4$. Note that when $0 < r < 4$, $r^2 < 16$ and $\frac{32}{r^2} > \frac{32}{16} = 2$, so $P'(r) = 2 - \frac{32}{r^2} < 0$. Similarly, when $r > 4$, $r^2 > 16$ and $\frac{32}{r^2} < \frac{32}{16} = 2$, so $P'(r) = 2 - \frac{32}{r^2} > 0$. It follows that $P(r)$ is decreasing for $0 < r < 4$ and increasing for $r > 4$. It follows that $P(4) = 2 \cdot 4 + \frac{32}{4} = 8 + 8 = 16$ is the minimum value of the perimeter function.

Thus the minimum possible perimeter of a pie slice with area of 16 cm^2 is 16 cm. ■

6. A small stone is dropped into a still pool, creating a circular ripple that moves outward from the point of impact at a constant rate. How is the area enclosed by the ripple changing after 2 s if the circumference of the ripple is changing at a rate of 2π m/s at this instant?

SOLUTION. Here is a crude sketch of the given situation:



The circumference of the circular ripple when it has radius r is $c = 2\pi r$ and we are told that $\left.\frac{dc}{dt}\right|_{t=2} = 2\pi\ m/s$. On the other hand, we must have $\frac{dc}{dt} = \frac{d}{dt}2\pi r = 2\pi\frac{dr}{dt}$, so $\frac{dr}{dt} = \frac{1}{2\pi}\cdot\frac{dc}{dt}$, and we are told that $\frac{dr}{dt}$, the rate at which the circular ripple moves outward from its centre, is constant. It follows that

$$\frac{dr}{dt} = \left.\frac{dr}{dt}\right|_{t=2} = \frac{1}{2\pi}\cdot\left.\frac{dc}{dt}\right|_{t=2} = \frac{1}{2\pi}\cdot 2\pi = 1\ m/s.$$

Since the radius of the circular ripple is 0 at time $t = 0\ s$, it then follows that $r = (2\ m)(1\ m/s) = 2\ m$ at time $t = 2\ s$.

The area enclosed by the circular ripple when it has radius r is $A = \pi r^2$, so at any given instant, with some help from the Chain Rule,

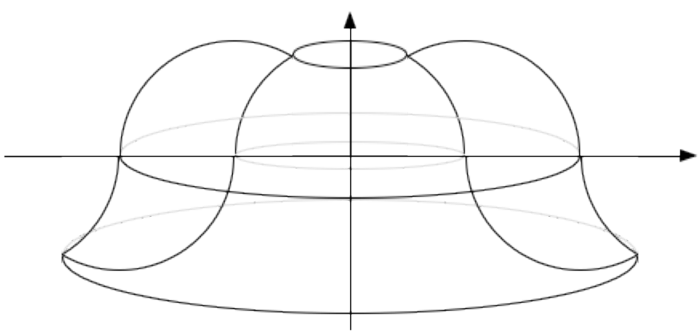
$$\frac{dA}{dt} = \frac{d}{dt}\pi r^2 = \left(\frac{d}{dr}\pi r^2\right)\cdot\frac{dr}{dt} = 2\pi r\frac{dr}{dt}.$$

Since $\frac{dr}{dt} = 1\ m/s$ and $r = 2\ m$ when $t = 2\ s$, we can conclude that:

$$\left.\frac{dA}{dt}\right|_{t=2} = 2\pi(2\ m)(1\ m/s) = 4\pi\ m^2/s \quad \blacksquare$$

7. Sketch the solid obtained by revolving the region between $y = \cos(x)$ and $y = \sin(x)$, for $\frac{\pi}{4} \le x \le \frac{5\pi}{4}$, about the y-axis and find its volume.

SOLUTION. Here is a crude sketch of the solid.



Since we revolved the given region about the y-axis and are using cylindrical shells, we need to integrate with respect to x. The cylindrical shell at x will have radius $r = x - 0 = x$ and height $h = \sin(x) - \cos(x)$. (Notice that for $\frac{\pi}{4} \le x \le \frac{5\pi}{4}$, we have $\sin(x) \ge \cos(x)$

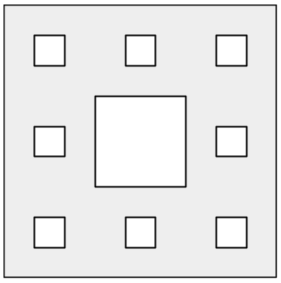
and that $\sin(x) = \cos(x)$ at the endpoints.) It follows that its volume is:

$$V = \int_{\pi/4}^{5\pi/4} 2\pi r h \, dx = 2\pi \int_{\pi/4}^{5\pi/4} x\left(\sin(x) - \cos(x)\right) \, dx$$

Use parts with $u = x$ and $v' = \sin(x) - \cos(x)$, so $u' = 1$ and $v = -\cos(x) - \sin(x)$.

$$\begin{aligned}
&= 2\pi x\left(-\cos(x) - \sin(x)\right)\Big|_{\pi/4}^{5\pi/4} - 2\pi \int_{\pi/4}^{5\pi/4} 1\left(-\cos(x) - \sin(x)\right) \, dx \\
&= 2\pi\frac{5\pi}{4}(-1)\left(\cos\left(\frac{5\pi}{4}\right) + \sin\left(\frac{5\pi}{4}\right)\right) - 2\pi\frac{\pi}{4}(-1)\left(\cos\left(\frac{\pi}{4}\right) + \sin\left(\frac{\pi}{4}\right)\right) \\
&\qquad + 2\pi \int_{\pi/4}^{5\pi/4} \left(\cos(x) + \sin(x)\right) \, dx \\
&= -\frac{5\pi^2}{2}\left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right) + \frac{\pi^2}{2}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) + 2\pi\left(\sin(x) - \cos(x)\right)\Big|_{\pi/4}^{5\pi/4} \\
&= \frac{6\pi^2}{2} \cdot \frac{2}{\sqrt{2}} + 2\pi\left(\cos\left(\frac{5\pi}{4}\right) - \sin\left(\frac{5\pi}{4}\right)\right) - 2\pi\left(\cos\left(\frac{\pi}{4}\right) - \sin\left(\frac{\pi}{4}\right)\right) \\
&= \frac{6\pi^2}{\sqrt{2}} + 2\pi 0 - 2\pi 0 = 3\sqrt{2}\pi^2 \qquad \blacksquare
\end{aligned}$$

8. Suppose we start with a unit square. In step 1 we divide it into $3 \times 3 = 9$ subsquares and then remove the middle one. In step 2 we divide each of the eight remaining subsquares into $3 \times 3 = 9$ smaller subsquares and remove the middle one in each case. (The picture at right is what you have after step 2.) At each step $n \geq 3$, we subdivide the remaining subsquares into $3 \times 3 = 9$ even smaller subsquares and remove the middle one in each case. What is the area of the object remaining after infinitely many steps?



Solution. In the beginning, let's call it step 0, we have a unit square, which has area 1. In step 1 we remove $\frac{1}{9}$ of this square, leaving us with an area of $\frac{8}{9}$. In step 2, we remove $\frac{1}{9}$ of each of the remaining subsquares, leaving us with $\frac{8}{9}$ of the area of $\frac{8}{9}$ we were left with from the step before. Thus we have an area of $\left(\frac{8}{9}\right)^2$ remaining after step 2. In general, given that we have an area of $\left(\frac{8}{9}\right)^n$ remaining after step n, we remove $\frac{1}{9}$ of that area in step $n+1$, leaving us with an area of $\left(\frac{8}{9}\right)^n\left(\frac{8}{9}\right) = \left(\frac{8}{9}\right)^{n+1}$.

The area A of the shape remaining after infinitely many steps is the limit as $n \to \infty$ of the area of the shape after n steps. Since $\frac{8}{9} < 1$, we have that $A = \lim_{n\to\infty}\left(\frac{8}{9}\right)^n = 0$.

How can we be sure the limit is correct? Suppose we are given an $\varepsilon > 0$; we may assume that $\varepsilon < 1$ as well. (Why?) Then:

$$\left|\left(\frac{8}{9}\right)^n - 0\right| < \varepsilon \iff \left(\frac{8}{9}\right)^n < \varepsilon \iff n\ln\left(\frac{8}{9}\right) = \ln\left(\left(\frac{8}{9}\right)^n\right) < \ln(\varepsilon) \iff n > \frac{\ln(\varepsilon)}{\ln\left(\frac{8}{9}\right)}$$

Note that since $\frac{8}{9} < 1$ and $\varepsilon < 1$, both $\ln\left(\frac{8}{9}\right)$ and $\ln(\varepsilon)$ are negative. Multiplying by the negative quantity $\ln\left(\frac{8}{9}\right)$ at the last step above reversed the inequality; also, since it is a quotient of negative numbers, the quantity $\frac{\ln(\varepsilon)}{\ln\left(\frac{8}{9}\right)}$ is positive. In any case, since every step is reversible, if we let N be any integer larger than $\frac{\ln(\varepsilon)}{\ln\left(\frac{8}{9}\right)}$, then whenever $n \geq N$, we will have $\left|\left(\frac{8}{9}\right)^n - 0\right| < \varepsilon$. Thus by the ε–N definition of limits at infinity, it follows that $\lim_{n\to\infty}\left(\frac{8}{9}\right)^n = 0$. ■

[Total = 100]

Part C. *Bonus problems!* If you feel like it, do one or both of these.

9. Recall that an integer greater than 1 is a prime number if it is not the product of two smaller positive integers. Determine whether or not the polynomial $p(x) = x^2 + x + 41$ always gives you prime numbers when $x \geq 0$ is an integer. *[1]*

SOLUTION. Amazingly, $p(n)$ is a prime number for each integer n from 0 through 40. However, $p(41) = 41^2 + 41 + 41 = 41(41 + 1 + 1) = 41 \cdot 43$ is not a prime number, so the polynomial does not always output prime numbers for non-negative integer inputs. ■

10. Write an original poem touching on calculus or mathematics in general. *[1]*

SOLUTION. You're on your own on this one! :-) ■

I HOPE THAT YOU ENJOYED THE COURSE. HAVE A GOOD BREAK!