Mathematics 1110H – Calculus I: Limits, derivatives, and Integrals
TRENT UNIVERSITY, Fall 2020

Solutions to Assignment #6
Solving Equations
Due on Wednesday, 9 December.

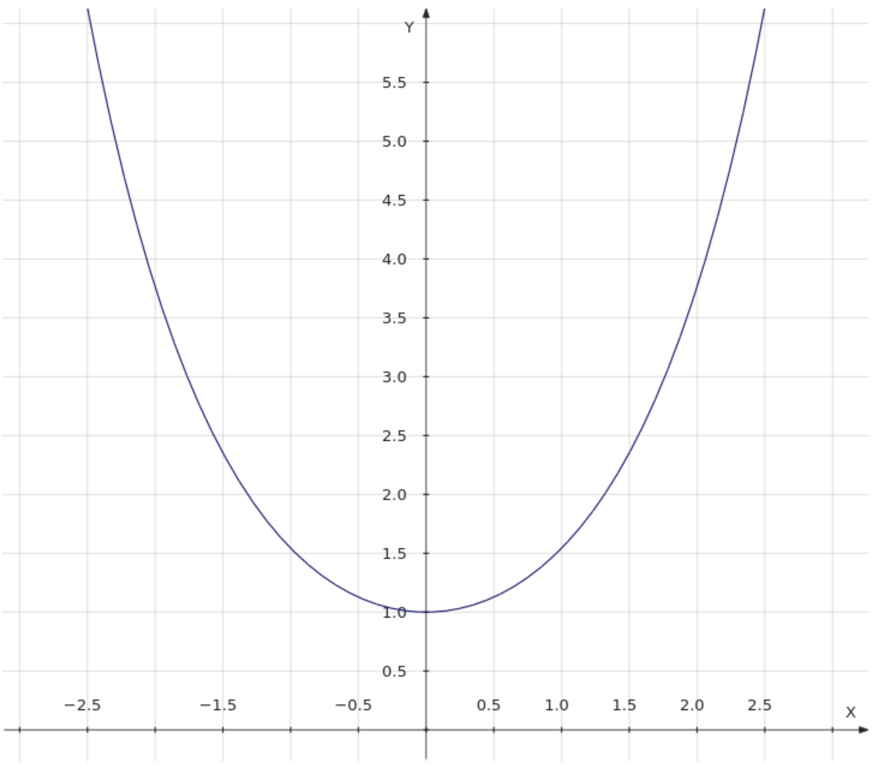
Recall that the hyperbolic cosine function is given by $\cosh(x) = \dfrac{e^x + e^{-x}}{2}$ for all real numbers x. Your task in this assignment will be to find its inverse function $\operatorname{arccosh}(x)$. Since $\cosh(x)$ is an even function, *i.e.* $\cosh(x) = \cosh(-x)$ for all x, we cannot hope to invert it over its entire domain, so we will seek to invert it only for $x \geq 0$, for which values of x $\cosh(x)$ is 1–1.

1. Derive an expression for $\operatorname{arccosh}(x)$ in terms of powers, roots, and the natural logarithm function. When does this expression make sense? *[6]*

NOTE. If we invert the part of $\cosh(x)$ for $x \geq 0$, the resulting function can only have output ≥ 0.

Hint: $y = \operatorname{arccosh}(x)$ exactly when $x = \cosh(y) = \dfrac{e^y + e^{-y}}{2}$. Solve the latter equation for y in terms of x. The quadratic formula is likely to be useful ...

SOLUTION. First, let's take a look at the graph of $\cosh(x)$:



The part of $\cosh(x)$ for which $x \geq 0$ starts at $y = 1$ for $x = 0$ and goes up from there. It follows that the inverse of this part will be defined only for $x \geq 1$, with $y = 0$ at $x = 1$, and will only have positive y-values for $x > 1$. This will let us do a sanity check on whatever expression we eventually obtain for the inverse.

We will now follow the hint and see what we get:

$$\begin{aligned}
y = \operatorname{arccosh}(x) &\iff x = \cosh(y) = \frac{e^y + e^{-y}}{2} \iff 2x = e^y + e^{-y} \\
&\iff 2xe^y = (e^y)^2 + e^{-y}e^y = (e^y)^2 + e^0 = (e^y)^2 + 1 \\
&\iff (e^y)^2 - 2xe^y + 1 = 0 \quad \text{which is a quadratic equation in } e^y \text{, so } \ldots \\
&\iff e^y = \frac{-(-2x) \pm \sqrt{(-2x)^2 - 4 \cdot 1 \cdot 1}}{2 \cdot 1} \\
&\iff e^y = \frac{2x \pm \sqrt{4x^2 - 4}}{2} = \frac{2x \pm 2\sqrt{x^2 - 1}}{2} = x \pm \sqrt{x^2 - 1} \\
&\iff y = \ln\left(x \pm \sqrt{x^2 - 1}\right)
\end{aligned}$$

Note first that $\sqrt{x^2 - 1}$ is defined only when $x^2 \geq 1$, *i.e.* when $|x| \geq 1$. From the discussion after looking at the graph of $\cosh(x)$, it follows that we can stick to $x \geq 1$ and need not worry about the possibility that $x \leq -1$. Second, note that when $x \geq 1$, we have that $x - \sqrt{x^2 - 1} \leq 1$ [Why?], which would make $\ln\left(x - \sqrt{x^2 - 1}\right) \leq 0$; however, $x + \sqrt{x^2 - 1} \geq 1$, which would make $\ln\left(x + \sqrt{x^2 - 1}\right) \geq 0$. If we are inverting the part of $\cosh(x)$ for which $x \geq 0$, it follows that we should take the expression for the inverse that gives us non-negative outputs.

Thus the inverse function for the part of $\cosh(x)$ where $x \geq 0$ is

$$\operatorname{arccosh}(x) = \ln\left(x + \sqrt{x^2 - 1}\right),$$

which is defined for $x \geq 1$. □

2. Use `Maple` to find an expression for $\operatorname{arccosh}(x)$. Is this expression equivalent to the one you obtained in answering **1**? *[4]*

Hint: If using `Maple`'s worksheet mode, you'll want to look up the `solve` command.

Solution. We'll try using the `solve` command in `Maple`, *per* the hint:

```
> solve(x = (e^y+e^(-y))*(1/2), y)
```

$$RootOf\left(2x - e^{\_Z} - e^{-\_Z}\right)$$

The appearance of *RootOf* is a sign that `Maple` doesn't have enough information to fully solve the problem. The problem in this case is that `Maple` is interpreting `e` as an unknown rather than the base of the natural exponential and logarithm functions. The easiest way to work around this is to use `Maple`'s version of the natural exponential function, called `exp`.

```
> solve(x = (exp(y)+exp(-y))*(1/2), y)
```

$$\ln\left(x+\sqrt{x^2-1}\right),\ \ln\left(x-\sqrt{x^2-1}\right)$$

At this point, just as we did in solving the problem by hand, we would have to choose the alternative that inverts the part of $\cosh(x)$ we want.

Just for fun, here is how the same problem could be solved in `SageMath`, an open-source counterpart to `Maple` and `Mathematica`:

```
sage: var("y")
      y
sage: solve(x==(e^y + e^(-y))/2,y)
      [y == log(x - sqrt(x^2 - 1)), y == log(x + sqrt(x^2 - 1))]
```

This is not too different from solving the problem in `Maple`, but two quirks of `SageMath` are apparent here. First, if you are going to use a variable other than x in a symbolic computation, you have to tell `SageMath` that it is a variable. Second, `SageMath` uses `==` to represent equality of functions. On the other hand, `SageMath` does interpret `e` as the base of the natural exponential and logarithm functions. ■