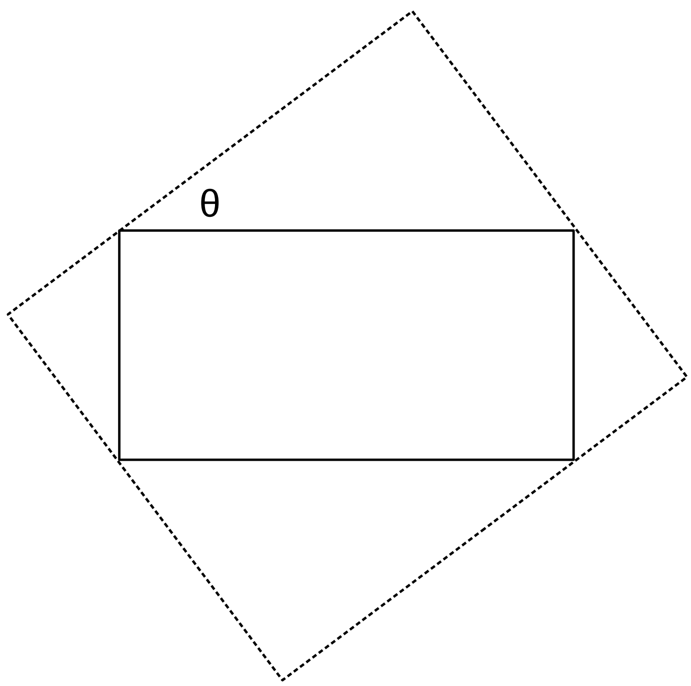
Mathematics 1110H – Calculus I: Limits, derivatives, and Integrals
Trent University, Fall 2020

Solution to Assignment #3
Rectangles Within/Without
Due on Friday, 23 October.

Submission: Scanned or photographed handwritten solutions are fine, so long as they are legible. Submission as a single pdf is strongly preferred, but other common formats are probably OK. (If not, we'll get back to you! :-) Please submit your solutions via Blackboard's Assignments module. If that fails, please email your solutions to the instructor at: `sbilaniuk@trentu.ca`

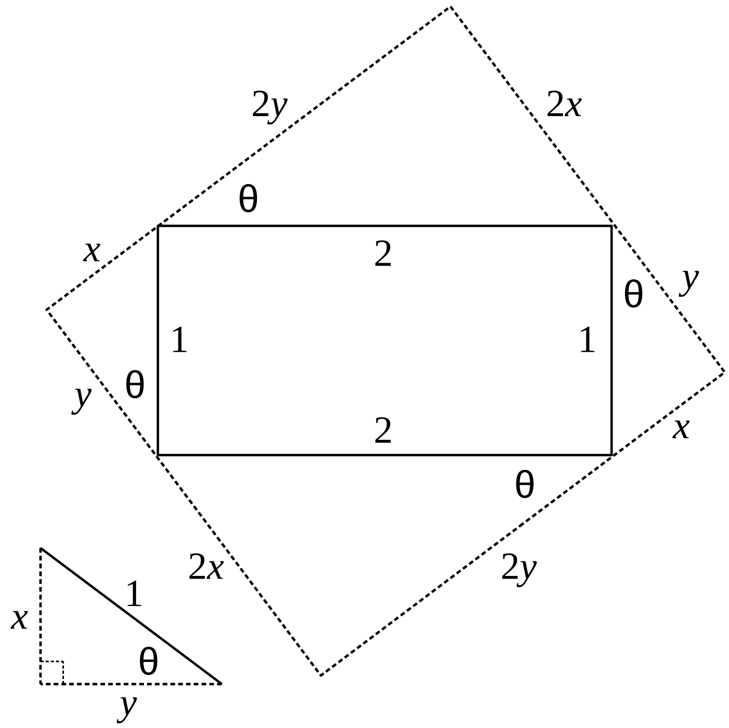
Rectangle A is *circumscribed* about rectangle B (and B is *inscribed* in A) if B is inside A and the corners of B touch the borders of A.

1. Suppose we are given a rectangle of height 1 and width 2. What is the maximum possible area of a rectangle circumscribed about the given one, as in the diagram below? *[10]*



Hint: Express the lengths of the sides of the circumscribed rectangle in terms of the angle θ between its sides and the sides of the given rectangle.

Solution. We will take the hint and express the lengths of the sides of the circumscribed rectangle in terms of the angle θ between its sides and the sides of the given 1×2 rectangle. Suppose we start with θ as in the diagram above. By symmetry, the angle at the opposite corner of the given rectangle between its other long side and the side of the circumscribed rectangle touching that corner is also θ. Since the sides of a rectangle meet at right angles at the corners, and this is true of both the given and circumscribed rectangles in the diagram, it is not hard to see that the angle θ also occurs in corresponding positions at the other two corners of the given rectangles, as in the diagram below. Note also that the least θ could be is 0 radians, in which case the circumscribed rectangle coincides with the given rectangle, and the most it could be is a right angle, *i.e.* $\frac{\pi}{2}$ radians, in which case the circumscribed rectangle again coincides with the given rectangle.



If one cuts the given rectangle out of the circumscribed rectangle, four right triangles are left behind. The two larger right triangles, which have the sides of length 2 of the given rectangle as their hypotenuses, are similar (*i.e.* have equal corresponding angles) to but have twice the linear dimensions of the two smaller right triangles that have the sides of length 1 of the given rectangle as their hypotenuses.

Consider one of the right triangles formed with a side of length 1 of the given rectangle as its hypotenuse, and let x be the length of its side opposite the angle θ and y be the length of its side adjacent to the angle θ. From basic trigonometry we know that $\sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{x}{1} = x$ and that $\cos(\theta) = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{y}{1} = y$. It follows that the circumscribed rectangle has length $x + 2y = \sin(\theta) + 2\cos(\theta)$ and width $2x + y = 2\sin(\theta) + \cos(\theta)$, being a bit arbitrary about which dimension is length and which is width. Thus the area of the circumscribed rectangle in terms of θ is:

$$\begin{aligned} A(\theta) &= \text{length} \times \text{width} = (x+2y)(2x+y) = (\sin(\theta)+2\cos(\theta))\,(2\sin(\theta)+\cos(\theta)) \\ &= 2\sin^2(\theta)+\sin(\theta)\cos(\theta)+4\cos(\theta)\sin(\theta)+2\cos^2(\theta) \\ &= 2\left(\sin^2(\theta)+\cos^2(\theta)\right)+5\cos(\theta)\sin(\theta) = 2\cdot 1+5\cdot\frac{1}{2}\sin(2\theta) = 2+\frac{5}{2}\sin(2\theta) \end{aligned}$$

Our task, therefore, boils down to finding the absolute maximum of $A(\theta) = 2 + \frac{5}{2}\sin(2\theta)$ for $0 \le \theta \le \frac{\pi}{2}$. As usual, we do this by comparing the values of $A(\theta)$ at the endpoints of the interval and at any critical points inside the interval.

$$A'(\theta) = \frac{d}{d\theta}\left(2+\frac{5}{2}\sin(2\theta)\right) = 0+\frac{5}{2}\cos(2\theta)\cdot\frac{d}{d\theta}(2\theta) = \frac{5}{2}\cos(2\theta)\cdot 2 = 5\cos(2\theta)$$

Thus $A'(\theta) = 5\cos(2\theta) = 0$ exactly when $\cos(2\theta) = 0$, which happens exactly when $2\theta = \frac{\pi}{2} + n\pi$ for some integer n, *i.e.* exactly when $\theta = \frac{\pi}{4} + \frac{n\pi}{2}$ for some integer n. The only point meeting these specifications between 0 and $\frac{\pi}{2}$, that is, the only critical point in our interval, is $\theta = \frac{\pi}{4}$.

Comparing values,

$$\begin{aligned} A(0) &= 2+\frac{5}{2}\sin(2\cdot 0) = 2+\frac{5}{2}\sin(0) = 2+\frac{5}{2}\cdot 0 = 2\,, \\ A\left(\frac{\pi}{4}\right) &= 2+\frac{5}{2}\sin\left(2\cdot\frac{\pi}{4}\right) = 2+\frac{5}{2}\cdot\sin\left(\frac{\pi}{2}\right) = 2+\frac{5}{2}\cdot 1 = \frac{9}{2} = 4.5\,, \\ \text{and } A\left(\frac{\pi}{2}\right) &= 2+\frac{5}{2}\sin\left(2\cdot\frac{\pi}{2}\right) = 2+\frac{5}{2}\cdot\sin(\pi) = 2+\frac{5}{2}\cdot 0 = 2\,, \end{aligned}$$

we see that the maximum of 4.5 occurs at the critical point $\theta = \frac{\pi}{4}$.

Thus 4.5 square units is the maximum area of a rectangle that can be circumscribed about a rectangle of width 1 and length 2. ■