Mathematics 1110H – Calculus I: Limits, derivatives, and Integrals
TRENT UNIVERSITY, Fall 2020

Assignment #1
Games With Limits
Due on Friday, 25 September.

The usual $\varepsilon - \delta$ definition of limits,

> DEFINITION. $\lim_{x \to a} f(x) = L$ exactly when for every $\varepsilon > 0$ there is a $\delta > 0$ such that for any x with $|x - a| < \delta$ we are guaranteed to have $|f(x) - L| < \varepsilon$ as well.

is pretty hard to wrap your head around the first time or three for most people. Here is less common definition, equivalent to the one above, which recasts the confusing logical structure of the above definition in terms of a game:

> ALTERNATE DEFINITION. The *limit game* for $f(x)$ at $x = a$ with target L is a three-move game played between two players A and B as follows:
>
> 1. A moves first, picking a small number $\varepsilon > 0$.
> 2. B moves second, picking another small number $\delta > 0$.
> 3. A moves third, picking an x that is within δ of a, *i.e.* $a - \delta < x < a + \delta$.
>
> To determine the winner, we evaluate $f(x)$. If it is within ε of the target L, *i.e.* $L - \varepsilon < f(x) < L + \varepsilon$, then player B wins; if not, then player A wins.
>
> With this idea in hand, $\lim_{x \to a} f(x) = L$ means that player B has a winning strategy in the limit game for $f(x)$ at $x = a$ with target L; that is, if B plays it right, B will win no matter what A tries to do. (Within the rules ... :-) Conversely, $\lim_{x \to a} f(x) \neq L$ means that player A is the one with a winning strategy in the limit game for $f(x)$ at $x = a$ with target L.

The game definition of limits isn't really better or worse that the usual $\varepsilon - \delta$ definition, but each is easier for some people to understand, and the exercise in trying it both ways usually helps in understanding what is really going on here.

1. Use one of these two definitions of limit to verify that $\lim_{x \to 3} (-2x + 7) = 1$. *[2]*

SOLUTION. We will use the standard ε–δ definition of limits. Thus, to verify that $\lim_{x \to 3} (-2x + 7) = 1$, we need to show that for any given $\varepsilon > 0$, one can find a $\delta > 0$ so that if $|x - 3| < \delta$, then $|(-2x + 7) - 1| < \varepsilon$. As usual, we try to reverse engineer the δ from the desired conclusion:

$$\begin{aligned} |(-2x + 7) - 1| < \varepsilon &\iff |-2x + 6| < \varepsilon \\ &\iff |-2(x - 3)| < \varepsilon \\ &\iff |-2| \cdot |x - 3| < \varepsilon \\ &\iff 2\,|x - 3| < \varepsilon \\ &\iff |x - 3| < \frac{\varepsilon}{2} \end{aligned}$$

It follows that $\delta = \frac{\varepsilon}{2}$ will do the job. Every step of the process above is fully reversible, so if $|x-3| < \delta = \frac{\varepsilon}{2}$, we can run through the process backwards to deduce that $|(-2x+7)-1| < \varepsilon$.

Since we can find a suitable δ for any ε in the definition of $\lim_{x \to 3} (-2x+7) = 1$, the limit is correct. $\square$

2. Use the definition of limit that you didn't use in answering question **1** to verify that $\lim_{x \to 2} x^2 \neq 0$. *[3]*

Solution. Since we used the standard ε–δ definition of limit in answering **1**, we will use the alternate (*i.e.* game-based) definition to answer this question. To verify that $\lim_{x \to 2} x^2 \neq 0$ according to the game version of the ε–δ definition we need to show that player A has a winning strategy. To develop one, let's review how the game works in this case:

1. A chooses an $\varepsilon > 0$.
2. B responds by choosing a $\delta > 0$.
3. A then picks an x with $2 - \delta < x < 2 + \delta$.

- A wins if $\left|x^2 - 0\right| \geq \varepsilon$, otherwise B wins.

Examining this sequence tells us that a winning strategy for A requires us to choose a $\varepsilon > 0$ such that no matter what $\delta > 0$ B chooses, there is some way to choose an x with $2 - \delta < x < 2 + \delta$, so that $\left|x^2 - 0\right| \geq \varepsilon$.

We choose an $\varepsilon > 0$ for A to play that is smaller than the difference between the alleged limit of 0 and what the function x^2 really achieves at $x = 2$, which is $2^2 = 4$. Since $0 < 1 < 4 - 0 = 4$, $\varepsilon = 1$ is a convenient choice.

Now suppose B plays $\delta > 0$. We will have A play $x = 2 + \frac{\delta}{2}$. (Why? This falls into the required range of x's, $2 - \delta < x < 2 + \delta$, and is to the right of $x = 2$. Since x^2 is increasing near $x = 2$, this puts the corresponding point on the graph even farther away from the alleged limit of 0.) This wins for A because

$$\left|\left(2 + \frac{\delta}{2}\right)^2 - 0\right| = \left(2 + \frac{\delta}{2}\right)^2 = 2^2 + 2 \cdot \frac{\delta}{2} + \delta^2 = 4 + \delta + \delta^2 > 4 > 1 = \varepsilon,$$

as required.

Thus A has a winning strategy in the limit game: play $\varepsilon = 1$ on the first move, and then respond with $x = 2 + \frac{\delta}{2}$ after B plays any $\delta > 0$. It follows by the game version of the ε–δ definition of limits that $\lim_{x \to 2} x^2 \neq 0$. $\square$

3. Use either definition of limits above to verify that $\lim_{x \to 0} x^2 = 0$. *[2]*

Solution. We'll use the usual definition. Suppose that we are given a $\varepsilon > 0$. As usual, we attempt to reverse-engineer a suitable $\delta > 0$:

$$\left|x^2 - 0\right| < \varepsilon \iff \left|x^2\right| < \varepsilon \iff |x|^2 < \varepsilon \iff |x| < \sqrt{\varepsilon} \iff |x - 0| < \sqrt{\varepsilon}$$

Since every step of the process is fully reversible, it follows that $\delta = \sqrt{\varepsilon}$ works, and so $\lim_{x \to 0} x^2 = 0$. $\square$

4. Use either definition of limits above to verify that $\lim_{x \to 2} x^2 = 4$. *[3]*

Hint: The choice of δ in **4** will probably require some indirect reasoning. Pick some arbitrary smallish positive number for δ as a first cut. If it doesn't do the job, but x is at least that close, you'll have more information to help pin down the δ you really need.

SOLUTION. We will use the game version of the definition of limits. Since we wish to verify that $\lim_{x \to 2} x^2 = 4$, we need to find a winning strategy for player B in the corresponding limit game. Let's review this game:

1. A chooses an $\varepsilon > 0$.
2. B responds by choosing a $\delta > 0$.
3. A then picks an x with $2 - \delta < x < 2 + \delta$, *i.e.* such that $|x - 2| < \delta$.

- B wins if $\left|x^2 - 4\right| < \varepsilon$, otherwise A wins.

A winning strategy for player B therefore consists of specifying how, given some $\varepsilon > 0$, to select a $\delta > 0$ so that matter how player A chooses an x with $|x - 2| < \delta$, $\left|x^2 - 4\right| < \varepsilon$ for that x.

We try to reverse a suitable $\delta > 0$ from the desired winning condition, just as we do using the standard definition of limits.

$$\left|x^2 - 4\right| < \varepsilon \iff |(x-2)(x+2)| < \varepsilon \iff |x - 2| < \frac{\varepsilon}{|x+2|}$$

Unfortunately, we cannot use $\dfrac{\varepsilon}{|x+2|}$ as our δ: short of having access to a working time machine (those physics types are really failing us there :-), we cannot know the x that player A will play in response to player B's choice δ. We can, however, decide on a restriction for δ that will limit player A's ability to choose x enough to let us select a δ that will work.

Observe that the root of $x + 2$, namely -2, is a distance of 4 from the root of $x - 2$, namely 2. Suppose we pick an upper bound for δ that is less than this distance, say 1. Note that $\delta \le 1$ implies that if $|x - 2| < \delta$, then $|x - 2| < 1$. Let's consider what this means for $|x + 2|$, and hence for $\dfrac{\varepsilon}{|x+2|}$.

$$\begin{aligned} |x-2| < 1 &\iff -1 < x - 2 < 1 \iff -1 + 4 < x - 2 + 4 < 1 + 4 \\ &\iff 3 < x + 2 < 5 \iff 3 < |x+2| < 5 \quad [x + 2 > 0 \text{, so } x + 2 = |x+2|.] \\ &\iff \frac{1}{3} > \frac{1}{|x+2|} > \frac{1}{5} \iff \frac{\varepsilon}{3} > \frac{\varepsilon}{|x+2|} > \frac{\varepsilon}{5} \end{aligned}$$

So as long as $\delta \le 1$ and $\delta \le \frac{\varepsilon}{5}$ (and, of course, $\delta > 0$), it will do the job:

$$\begin{aligned} |x - 2| < \delta &\implies |x-2| < \frac{\varepsilon}{5} \quad \left[\text{Since } \delta \le \frac{\varepsilon}{5}.\right] \\ &\implies |x-2| < \frac{\varepsilon}{|x+2|} \quad \left[\text{Since } \frac{\varepsilon}{5} < \frac{\varepsilon}{|x+2|} \text{ because } \delta \le 1.\right] \\ &\implies \left|x^2 - 4\right| = |x-2| \cdot |x+2| < \varepsilon \end{aligned}$$

Selecting any δ with $0 < \delta \le \min\left(1, \frac{\varepsilon}{5}\right)$ therefore wins the game for player B.

Since a winning strategy exists for player B in the corresponding limit game. we have that $\lim_{x \to 2} x^2 = 4$. ■