Mathematics 1110H – Calculus I: Limits, derivatives, and Integrals
TRENT UNIVERSITY, Fall 2020

Solution to Quiz #6
Tuesday, 3 November .

Available on Blackboard from 12:00 a.m. on Tuesday, 3 November.
Due on Blackboard by 11:59 p.m. on Tuesday, 3 November.
Solutions will be posted on Thursday, 3 November.

Scans or photos of handwritten work are entirely acceptable so long as they are legible and in some common format; solutions submitted as a single pdf are preferred, if you can manage it. If you can't submit your solutions on time via Blackboard's Assignments module for some reason, please email them to the instructor at: `sbilaniuk@trentu.ca`

Reminder: Per the course outline, *all work submitted for credit must be written up entirely by yourself, giving due credit to all relevant sources of help and information.* For this quiz, you are permitted to use your textbook and all other course material, from this and any other mathematics course(s) you have taken or are taking now, but *you may not use any other sources or aids, nor give or receive any help*, except to ask the instructor to clarify questions and to use a calculator (any that you like).

1. Find any and all intercepts, vertical and horizontal asymptotes, intervals of increase and decrease, local maxima and minima, intervals of concavity, and inflection points, of $f(x) = xe^{-x^2}$ on the interval $(-\infty, \infty)$, and sketch its graph. *[5]*

SOLUTION. We run through the checklist. As this is the same function we saw on Quiz #5, some of what follows may seem familiar ... :-)

0° *Intercepts.* When $x = 0$, $f(0) = 0e^{-0^2} = 0 \cdot 1 = 0$, so the y-intercept is $y = 0$.

Since $e^t > 0$ for all real numbers t, $f(x) = xe^{-x^2} = 0$ exactly when $x = 0$, so the y-intercept is also the only x-intercept.

1° *Vertical Asymptotes.* Since $f(x) = xe^{-x^2}$ is defined and differentiable for all x, being a product and composition of functions which are defined and differentiable everywhere, it is also continuous everywhere and hence cannot have any vertical asymptotes.

2° *Horizontal Asymptotes.* We take the limit going to infinity in each direction and see what we get, with some help from l'Hôpital's Rule:

$$\lim_{x\to-\infty} xe^{-x^2} = \lim_{x\to-\infty} \frac{x}{e^{x^2}} \begin{matrix}\to -\infty \\ \to +\infty\end{matrix} = \lim_{x\to-\infty} \frac{\frac{d}{dx}x}{\frac{d}{dx}e^{x^2}} = \lim_{x\to-\infty} \frac{1}{2xe^{x^2}} \begin{matrix}\to 1 \\ \to -\infty\end{matrix} = 0^-$$

$$\lim_{x\to+\infty} xe^{-x^2} = \lim_{x\to+\infty} \frac{x}{e^{x^2}} \begin{matrix}\to +\infty \\ \to +\infty\end{matrix} = \lim_{x\to+\infty} \frac{\frac{d}{dx}x}{\frac{d}{dx}e^{x^2}} = \lim_{x\to+\infty} \frac{1}{2xe^{x^2}} \begin{matrix}\to 1 \\ \to +\infty\end{matrix} = 0^+$$

Thus $f(x)$ has a horizontal asymptote of $y = 0$ in both directions. Note that as $x \to -\infty$, $f(x) = xe^{-x^2}$ is negative and so approaches 0 from below, and as $x \to +\infty$, $f(x) = xe^{-x^2}$ is positive and so approaches 0 from above.

3° *Slopes.* (Increase/decrease and local max/min.) Recall from 1° that $f(x) = xe^{-x^2}$ is defined and differentiable for all x. We will use the derivative to find the critical points and intervals of increase and decrease of $f(x)$.

$$\begin{aligned} f'(x) &= \frac{d}{dx} xe^{-x^2} = \left[\frac{d}{dx}x\right] e^{-x^2} + x\left[\frac{d}{dx}e^{-x^2}\right] = 1e^{-x^2} + xe^{-x^2} \cdot \frac{d}{dx}\left(-x^2\right) \\ &= e^{-x^2} + xe^{-x^2} \cdot (-2x) = e^{-x^2} - 2x^2e^{-x^2} = \left(1-2x^2\right)e^{-x^2} \end{aligned}$$

Like $f(x)$ itself, $f'(x)$ is defined and differentiable, and hence continuous, for all x, being a product and composition of functions which are defined and differentiable everywhere. Since $e^t > 0$ for all real numbers t, $f'(x)$ is < 0, $= 0$, or is > 0, exactly when $1 - 2x^2$ is < 0, $= 0$, or is > 0, respectively. This happens when $x^2 > \frac{1}{2}$, $x^2 = \frac{1}{2}$, or $x^2 < \frac{1}{2}$, respectively, *i.e.* when $x < -\frac{1}{\sqrt{2}}$ or $x > \frac{1}{\sqrt{2}}$, $x = \pm\frac{1}{\sqrt{2}}$ (the critical points), or $-\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$, respectively. It follows that $f(x)$ is decreasing when $x < -\frac{1}{\sqrt{2}}$ or $x > \frac{1}{\sqrt{2}}$ and increasing when $-\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$, so the critical point $x = -\frac{1}{\sqrt{2}}$ is a local minimum and the critical point $x = \frac{1}{\sqrt{2}}$ is a local maximum.

As usual, we can summarize this in a table:

x	$\left(-\infty, -\frac{1}{\sqrt{2}}\right)$	$-\frac{1}{\sqrt{2}}$	$\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$	$\frac{1}{\sqrt{2}}$	$\left(\frac{1}{\sqrt{2}}, \infty\right)$
$f'(x)$	$-$	0	$+$	0	$-$
$f(x)$	$\downarrow$	min	$\uparrow$	max	$\downarrow$

4° *Curvature.* (Concavity and inflection points.) As noted above, $f'(x)$ is defined and differentiable for all x. We will use $f''(x)$, the derivative of $f'(x)$, to determine where $f(x)$ is concave up and where it is concave down, and also find any inflection points.

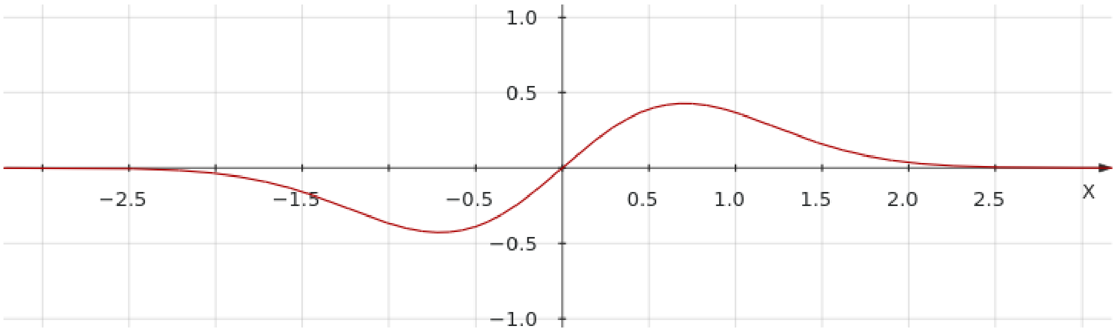
$$\begin{aligned} f''(x) &= \frac{d}{dx}\left(1-2x^2\right)e^{-x^2} = \left[\frac{d}{dx}\left(1-2x^2\right)\right]e^{-x^2} + \left(1-2x^2\right)\left[\frac{d}{dx}e^{-x^2}\right] \\ &= (-4x)e^{-x^2} + \left(1-2x^2\right)e^{-x^2} \cdot \frac{d}{dx}\left(-x^2\right) = -4xe^{-x^2} + \left(1-2x^2\right)e^{-x^2} \cdot (-2x) \\ &= -4xe^{-x^2} + \left(-2x+4x^3\right)e^{-x^2} = \left(4x^3-6x\right)e^{-x^2} = 2x\left(2x^2-3\right)e^{-x^2} \end{aligned}$$

Since $e^t > 0$ for all real numbers t, $f''(x) = 2x\left(2x^2-3\right)e^{-x^2}$ exactly when $2x\left(2x^2-3\right) = 0$, *i.e.* exactly when $x = 0$ or $x = \pm\sqrt{3/2}$. When $x < -\sqrt{3/2}$, $x < 0$ and $2x^2 - 3 > 0$, so $f''(x) < 0$ and $f(x)$ is concave down; when $-\sqrt{3/2} < x < 0$, then $x < 0$ and $2x^2 - 3 < 0$, so $f''(x) > 0$ and $f(x)$ is concave up; when $0 < x < \sqrt{3/2}$, then $x > 0$ and $2x^2 - 3 < 0$, so $f''(x) < 0$ and $f(x)$ is concave down; and when $x > \sqrt{3/2}$, then $x > 0$ and $2x^2 - 3 > 0$, so $f''(x) > 0$ and $f(x)$ is concave up. It follows that each of $x = 0$ and $x = \pm\sqrt{3/2}$ is an inflection point.

As usual, we can summarize this in a table:

x	$\left(-\infty, -\sqrt{3/2}\right)$	$-\sqrt{3/2}$	$\left(-\sqrt{3/2}, 0\right)$	0	$\left(0, \sqrt{3/2}\right)$	$\sqrt{3/2}$	$\left(\sqrt{3/2}, \infty\right)$
$f''(x)$	$-$	0	$+$	0	$-$	0	$+$
$f(x)$	$\frown$	infl	$\smile$	infl	$\frown$	infl	$\smile$

5° *Sketch.* Cheating a little bit, here is the graph of $f(x) = xe^{-x^2}$, as drawn by a program called `KmPlot`:



And that's all, folks! ■