Mathematics 1110H – Calculus I: Limits, derivatives, and Integrals
TRENT UNIVERSITY, Fall 2020

Solution to Quiz #5
Tuesday, 20 October .

1. Find the absolute maxima and minima, if any, of $f(x) = xe^{-x^2}$ on the interval $[0, \infty)$. *[5]*

SOLUTION. First, since $f(x) = xe^{-x^2}$ is a product and composition of functions which are defined and differentiable for all x, it cannot have any vertical asymptotes.

Second, since the interval is infinite in the positive direction, we check to see how the function behaves as $x \to \infty$ with the help of l'Hôpital's Rule:

$$\lim_{x\to\infty} xe^{-x^2} = \lim_{x\to\infty} \frac{x}{e^{x^2}} \begin{matrix} \to \infty \\ \to \infty \end{matrix} = \lim_{x\to\infty} \frac{\frac{d}{dx}x}{\frac{d}{dx}e^{x^2}} = \lim_{x\to\infty} \frac{1}{2xe^{x^2}} \begin{matrix} \to 1 \\ \to \infty \end{matrix} = 0^+$$

Note that as $x \to \infty$, $f(x) = xe^{-x^2}$ is positive and so approaches 0 from above.

Third, at the left endpoint of the interval we have $f(0) = 0e^{-0^2} = 0 \cdot 1 = 0$.

Fourth, we check for local maxima and minima inside the interval. Note that any such must occur at critical points because $f(x)$ is defined and differentiable for all x.

$$f'(x) = \frac{d}{dx}xe^{-x^2} = 1 \cdot e^{-x^2} + x \cdot e^{-x^2}(-2x) = \left(1 - 2x^2\right) e^{-x^2}$$

Since $e^t > 0$ for all $t \in \mathbb{R}$, $f'(x) = 0$ exactly when $1 - 2x^2 = 0$, *i.e.* when $x = \pm\frac{1}{\sqrt{2}}$. $x = -\frac{1}{\sqrt{2}}$ is not in the interval $[0, \infty)$, but $x = +\frac{1}{\sqrt{2}}$ is. Since $1 - 2x^2$ (and hence $f'(x)$) is > 0 when $0 < x < \frac{1}{\sqrt{2}}$ and is < 0 when $x > \frac{1}{\sqrt{2}}$, $f(x)$ is increasing from $x = 0$ to $x = \frac{1}{\sqrt{2}}$ and decreasing from $x = \frac{1}{\sqrt{2}}$ out to ∞. Thus $f(x)$ has a local maximum of

$$f\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}e^{-(1/\sqrt{2})^2} = \frac{1}{\sqrt{2}}e^{-1/2} = \frac{1}{e^{1/2}\sqrt{2}} = \frac{1}{\sqrt{2e}}$$

at $x = \frac{1}{\sqrt{2}}$.

Comparing the values and behaviour of the function obtained above, it is clear that the function has an absolute minimum on the given interval of 0 at $x = 0$ and an absolute maximum of $\frac{1}{\sqrt{2e}}$ at $x = \frac{1}{\sqrt{2}}$. ■