Mathematics 1110H – Calculus I: Limits, Derivatives, and Integrals
TRENT UNIVERSITY, Fall 2020
Solutions to the Take-Home Final Examination for Section A

Available on Blackboard from 12:00 a.m. on Tuesday, 15 December.
Due on Blackboard by 11:59 p.m. on Thursday, 17 December.

Submission: Scans or photos of handwritten work are entirely acceptable so long as they are legible and in some common format; solutions submitted as a single pdf are strongly preferred. If submission via Blackboard's Assignments module fails repeatedly, then (only as a *last* resort) email them to the instructor at: `sbilaniuk@trentu.ca`

Allowed aids: For this exam, you are permitted to use your textbook and all other course material, from this and any other mathematics course(s) you have taken or are taking now, but *you may not use any other sources or aids, nor give or receive any help*, except to ask the instructor to clarify questions and to use a calculator (any that you like).

Instructions: Do parts **U** and **V**, and, if you wish, part **W**. Please show all your work and justify all your answers. *If in doubt about something,* **ask!**

Part U. Do *all four* (4) of **1**–**4**. *[Subtotal = 74]*

1. Compute $\frac{dy}{dx}$ as best you can in any *five* (5) of **a**–**f**. *[20 = 5 × 4 each]*

a $y = \frac{e^x}{x+1}$ **b.** $y = \int_0^{\ln(x)} e^{2t}\,dt$ **c.** $y = e^{-x^2}$

d. $y = 3^{\sin(x)}$ **e.** $\ln\left(x^2+y^2\right) = 0$ **f.** $y = \sqrt{x\tan(x)}$

SOLUTIONS. **a.** *Quotient Rule.*

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{e^x}{x+1}\right) = \frac{\left(\frac{d}{dx}e^x\right)(x+1) - e^x\left(\frac{d}{dx}(x+1)\right)}{(x+1)^2} = \frac{e^x(x+1) - e^x \cdot 1}{(x+1)^2}$$
$$= \frac{xe^x + e^x - e^x}{(x+1)^2} = \frac{xe^x}{(x+1)^2} \qquad \square$$

b. *Fundamental Theorem of Calculus and the Chain Rule.*

$$\frac{dy}{dx} = \frac{d}{dx}\left(\int_0^{\ln(x)} e^{2t}\,dt\right) = e^{2\ln(x)} \cdot \frac{d}{dx}\ln(x) = \left(e^{\ln(x)}\right)^2 \cdot \frac{1}{x} = x^2 \cdot \frac{1}{x} = x \qquad \square$$

c. *Chain Rule and Power Rule.*

$$\frac{dy}{dx} = \frac{d}{dx}e^{-x^2} = e^{-x^2} \cdot \frac{d}{dx}\left(-x^2\right) = e^{-x^2} \cdot (-2x) = -2xe^{-x^2} \qquad \square$$

d. *Exponential derivative formula and the Chain Rule.*

$$\frac{dy}{dx} = \frac{d}{dx}3^{\sin(x)} = \ln(3) \cdot 3^{\sin(x)} \cdot \frac{d}{dx}\sin(x) = \ln(3) \cdot 3^{\sin(x)} \cdot \cos(x)$$
$$= \ln(3)\cos(x)\,3^{\sin(x)} \qquad \square$$

d. $3 = e^{\ln(3)}$ *and the Chain Rule.*

$$\frac{dy}{dx} = \frac{d}{dx}3^{\sin(x)} = \frac{d}{dx}\left(e^{\ln(3)}\right)^{\sin(x)} = \frac{d}{dx}e^{\ln(3)\sin(x)} = e^{\ln(3)\sin(x)} \cdot \frac{d}{dx}\ln(3)\sin(x)$$
$$= \left(e^{\ln(3)}\right)^{\sin(x)} \ln(3)\cos(x) = \ln(3)\cos(x)\, 3^{\sin(x)} \qquad \square$$

e. *Implicit differentiation and the Chain Rule*

$$\begin{aligned}\frac{d}{dx}\ln\left(x^2+y^2\right) = \frac{d}{dx}0 &\implies \frac{1}{x^2+y^2} \cdot \frac{d}{dx}\left(x^2+y^2\right) = 0\\ &\implies \frac{1}{x^2+y^2} \cdot \left(2x + 2y \cdot \frac{dy}{dx}\right) = 0\\ &\implies 2x + 2y\frac{dy}{dx} = 0\left(x^2+y^2\right) = 0\\ &\implies \frac{dy}{dx} = \frac{-2x}{2y} = -\frac{x}{y} \qquad \square\end{aligned}$$

e. *Some preliminary algebra, implicit differentiation, and the Chain Rule*

$$\begin{aligned}\ln\left(x^2+y^2\right) &\implies x^2+y^2 = 1\\ &\implies \frac{d}{dx}\left(x^2+y^2\right) = \frac{d}{dx}1\\ &\implies 2x + 2y \cdot \frac{dy}{dx} = 0\\ \implies \frac{dy}{dx} &= \frac{-2x}{2y} = -\frac{x}{y} \qquad \square\end{aligned}$$

e. *More preliminary algebra and the Chain Rule.*

$$\begin{aligned}\ln\left(x^2+y^2\right) &\implies x^2+y^2 = 1 \implies y = \pm\sqrt{1-x^2}\\ &\implies \frac{dy}{dx} = \frac{d}{dx}\left(\pm\sqrt{1-x^2}\right) = \frac{\pm 1}{2\sqrt{1-x^2}} \cdot \frac{d}{dx}\left(1-x^2\right)\\ &= \frac{\pm 1}{2\sqrt{1-x^2}} \cdot (-2x) = \frac{\mp x}{\sqrt{1-x^2}} \qquad \square\end{aligned}$$

f. *Chain Rule and Product Rule.*

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\sqrt{x\tan(x)} = \frac{1}{2\sqrt{x\tan(x)}} \cdot \frac{d}{dx}\left(x\tan(x)\right)\\ &= \frac{1}{2\sqrt{x\tan(x)}} \cdot \left(\left(\frac{d}{dx}x\right)\tan(x) + x\frac{d}{dx}\tan(x)\right)\\ &= \frac{1}{2\sqrt{x\tan(x)}} \cdot \left(1 \cdot \tan(x) + x \cdot \sec^2(x)\right) = \frac{\tan(x) + x\sec^2(x)}{2\sqrt{x\tan(x)}} \qquad \blacksquare\end{aligned}$$

2. Evaluate any *five* (5) of the integrals **a**–**f**. *[20 = 5 × 4 each]*

a. $\displaystyle\int xe^{x-1}\,dx$ **b.** $\displaystyle\int_0^{\pi/8} \tan(2y)\,dy$ **c.** $\displaystyle\int e^z \ln\left((e^z+1)^2\right)\,dz$

d. $\displaystyle\int_0^3 (w-1)(w+3)\,dw$ **e.** $\displaystyle\int \frac{v+\arctan(v)}{1+v^2}\,dv$ **f.** $\displaystyle\int_0^2 \frac{4u}{\sqrt{4+u^2}}\,du$

SOLUTIONS. **a.** *Substitution and integration by parts.* We will use the substitution $w = x-1$, so $dw = dx$ and $x = w+1$, followed by the parts $u = w+1$ and $v' = e^w$, so $u' = 1$ and $v = e^w$.

$$\int xe^{x-1}\,dx = \int (w+1)e^w\,dw = (w+1)e^w - \int 1\cdot e^w\,dw = (w+1)e^w - e^w + C$$
$$= xe^{x-1} - e^{x-1} + C = (x-1)e^{x-1} + C \qquad \square$$

a. *A bit of algebra and integration by parts.* We will use the parts $u = x$ abd $v' = e^x$, so $u' = 1$ and $v = e^x$.

$$\int xe^{x-1}\,dx = \int xe^x e^{-1}\,dx = e^{-1}\int xe^x\,dx = e^{-1}\left[xe^x - \int 1\cdot e^x\,dx\right]$$
$$= e^{-1}\left[xe^x - e^x\right] + C = e^{-1}(x-1)e^x + C = (x-1)e^{x-1} + C \qquad \square$$

b. *Substitution and an integral formula.* We will use the substitution $w = 2y$, so $dw = 2\,dy$ and $dy = \frac{1}{2}\,dw$, and change the limits as we go along: $\begin{matrix} y & 0 & \pi/8 \\ w & 0 & \pi/4 \end{matrix}$.

$$\int_0^{\pi/8} \tan(2y)\,dy = \int_0^{\pi/4} \tan(w)\,\frac{1}{2}\,dw = \frac{1}{2}\int_0^{\pi/4} \tan(w)\,dw = -\frac{1}{2}\ln\left(\cos(w)\right)\Big|_0^{\pi/4}$$
$$= \left[-\frac{1}{2}\ln\left(\cos(\pi/4)\right)\right] - \left[-\frac{1}{2}\ln\left(\cos(0)\right)\right] = -\frac{1}{2}\ln\left(\frac{1}{\sqrt{2}}\right) + \frac{1}{2}\ln(1)$$
$$= -\frac{1}{2}\ln\left(2^{-1/2}\right) + \frac{1}{2}\cdot 0 = -\frac{1}{2}\left(-\frac{1}{2}\right)\ln(2) = \frac{1}{4}\ln(2) \qquad \square$$

b. *A bit of trigonometry and substitution.* We will use the fact that $\tan(x) = \dfrac{\sin(x)}{\cos(x)}$, followed by the substitution $u = \cos(2y)$, so $du = -2\sin(2y)\,dy$ and $\sin(2y)\,dy = \left(-\frac{1}{2}\right)\,du$. We will also change the limits as we go along: $\begin{matrix} y & 0 & \pi/8 \\ u & 1 & 1/\sqrt{2} \end{matrix}$.

$$\int_0^{\pi/8} \tan(2y)\,dy = \int_0^{\pi/8} \frac{\sin(2y)}{\cos(2y)}\,dy = \int_1^{1/\sqrt{2}} \frac{1}{u}\left(-\frac{1}{2}\right)\,du = -\frac{1}{2}\int_1^{1/\sqrt{2}} \frac{1}{u}\,du$$
$$= \frac{1}{2}\int_{1/\sqrt{2}}^1 \frac{1}{u}\,du = \frac{1}{2}\ln(u)\Big|_{1/\sqrt{2}}^1 = \frac{1}{2}\ln(1) - \frac{1}{2}\ln\left(\frac{1}{\sqrt{2}}\right)$$
$$= \frac{1}{2}\cdot 0 - \frac{1}{2}\ln\left(2^{-1/2}\right) = 0 - \frac{1}{2}\left(-\frac{1}{2}\right)\ln(2) = \frac{1}{4}\ln(2) \qquad \square$$

c. *Substitution, a teensy bit of algebra, and integration by parts.* We will use the substitution $w = e^z + 1$, so $dw = e^z\,dz$, and later on integration by parts with $u = \ln(w)$ and $v' = 2$, so $u' = 1/w$ and $v = 2w$.

$$\int e^z \ln\left(\left(e^z+1\right)^2\right)\,dz = \int \ln\left(w^2\right)\,dw = \int 2\ln(w)\,dw = 2w\ln(w) - \int \frac{1}{w}\cdot 2w\,dw$$
$$= 2w\ln(w) - \int 2\,dw = 2w\ln(w) - 2w + C$$
$$= 2\left(e^z+1\right)\ln\left(e^z+1\right) - 2\left(e^z+1\right) + C \qquad \square$$

d. *A bit of algebra and the Sum and Power Rules.*

$$\int_0^3 (w-1)(w+3)\,dw = \int_0^3 \left(w^2+2w-3\right)\,dw = \left.\left(\frac{w^3}{3} + 2\frac{w^2}{2} - 3w\right)\right|_0^3$$
$$= \left(\frac{3^3}{3} + 3^2 - 3\cdot 3\right) - \left(\frac{0^3}{3} + 0^3 - 3\cdot 0\right)$$
$$= (9+9-9) - (0+0-0) = 9 \qquad \square$$

e. *Sum Rule and two different substitutions.* We will separate the integral into two parts and then use different substitutions in each part. In one part we will use the substitution $s = 1+v^2$, so $ds = 2v\,dv$ and $v\,dv = \frac{1}{2}\,ds$, and in the other part we will use the substitution $t = \arctan(v)$, so $dt = \dfrac{1}{1+v^2}\,dv$.

$$\int \frac{v+\arctan(v)}{1+v^2}\,dv = \int \frac{v}{1+v^2}\,dv + \int \frac{\arctan(v)}{1+v^2}\,dv = \int \frac{1}{s}\cdot\frac{1}{2}\,ds + \int t\,dt$$
$$= \frac{1}{2}\ln(s) + \frac{t^2}{2} + C = \frac{1}{2}\ln\left(1+v^2\right) + \frac{1}{2}\arctan^2(v) + C \qquad \square$$

f. *Substitution and the Power Rule.* We will use the substitution $w = 4+u^2$, so $dw = 2u\,du$, and change the limits as we go along:

u	0	2
w	4	8

$$\int_0^2 \frac{4u}{\sqrt{4+u^2}}\,du = \int_0^2 \frac{2}{\sqrt{4+u^2}}\,2u\,du = \int_4^8 \frac{2}{\sqrt{w}}\,dw = \int_4^8 2w^{-1/2}\,dw$$
$$= 2\cdot\left.\frac{w^{1/2}}{1/2}\right|_4^8 = 4\sqrt{w}\Big|_4^8 = 4\sqrt{8} - 4\sqrt{4} = 8\sqrt{2} - 8 \qquad \square$$

f. *A different substitution and the Power Rule.* This time we will use the substitution $z = \sqrt{4+u^2}$, so $dz = \dfrac{2u}{2\sqrt{4+u^2}}\,du = \dfrac{u}{\sqrt{4+u^2}}\,du$, and change the limits as we go along:

u	0	2
z	2	$2\sqrt{2}$

$$\int_0^2 \frac{4u}{\sqrt{4+u^2}}\,du = \int_2^{2\sqrt{2}} 4\,dz = 4z\Big|_2^{2\sqrt{2}} = 4\cdot 2\sqrt{2} - 4\cdot 2 = 8\sqrt{2} - 8 \qquad \blacksquare$$

3. Do any *five* (5) of **a**–**g**. *[20 = 5 × 4 each]*

a. Consider the parametric curve given by $x = \cos(t)$ and $y = \sin(t)$ for $0 \le t \le \pi$. What is the slope of the tangent line to the curve when $t = \frac{\pi}{4}$?

b. Compute the area of the finite region between $y = \sqrt{x}$ and $y = \frac{x}{2}$.

c. Compute $\lim_{x\to\infty} e^{-x}\ln(x)$.

d. Use the limit definition of the derivative to show that $\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}$ for all $x > 0$.

e. Find the absolute minimum value of $f(x) = \arctan\left(x^2\right)$ on $(-\infty, \infty)$.

f. Use the ε–δ definition of limits to verify that $\lim_{x\to-2} x^2 = 4$.

g. Find the volume of the solid obtained by revolving the region below $y = 3$ and above $y = 4 - x$, for $1 \le x \le 3$ about the x-axis.

SOLUTIONS. **a.** *The respectable way.* Since $x^2+y^2 = \cos^2(t)+\sin^2(t) = 1$ for all points on the curve, it is a part of the unit circle centred at the origin. Moreover, since $\cos(t)$ changes from 1 to -1 and $\sin(t)$ changes from 0 to 1 (at $t = \pi/2$) and back to 0 as t changes from 0 to π, the given parameteric curve is the upper half of the unit circle $x^2 + y^2 = 1$, for which $y = \sqrt{1-x^2}$. At $t = \pi/4$, we have $x = \cos(\pi/4) = 1/\sqrt{2}$ and $y = \sin(\pi/4) = 1/\sqrt{2}$.

All this means that we need to find the slope of the tangent line to $y = \sqrt{1-x^2}$ at $x = 1/\sqrt{2}$. The slope at x is given by

$$\frac{dy}{dx} = \frac{d}{dx}\sqrt{1-x^2} = \frac{1}{2\sqrt{1-x^2}} \cdot \frac{d}{dx}\left(1-x^2\right) = \frac{1}{2\sqrt{1-x^2}} \cdot (-2x) = \frac{-x}{\sqrt{1-x^2}}\,.$$

It follows that the slope of the tangent line at the given point, when $x = 1/\sqrt{2}$, is

$$\left.\frac{dy}{dx}\right|_{x=1/\sqrt{2}} = \frac{-1/\sqrt{2}}{\sqrt{1-\left(1/\sqrt{2}\right)^2}} = \frac{-1/\sqrt{2}}{\sqrt{1-\frac{1}{2}}} = \frac{-1/\sqrt{2}}{\sqrt{1/2}} = \frac{-1/\sqrt{2}}{1/\sqrt{2}} = -1\,. \qquad \square$$

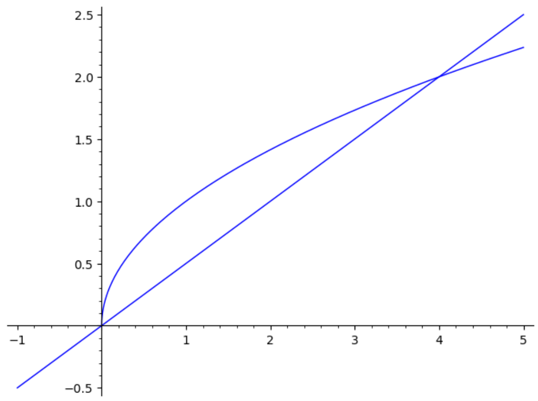
a. *The quick and dirty way.* We'll treat $\frac{dy}{dx}$ and such as fractions (and get away with it). The slope of the tangent line when $t = \pi/4$ is then given by:

$$\left.\frac{dy}{dx}\right|_{t=\pi/4} = \left.\frac{dy}{dt}\cdot\frac{dt}{dx}\right|_{t=\pi/4} = \left.\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right|_{t=\pi/4} = \left.\frac{\frac{d}{dt}\sin(t)}{\frac{d}{dt}\cos(t)}\right|_{t=\pi/4}$$

$$= \left.\frac{\cos(t)}{-\sin(t)}\right|_{t=\pi/4} = \frac{\cos(\pi/4)}{-\sin(\pi/4)} = \frac{1/\sqrt{2}}{-1/\sqrt{2}} = -1 \qquad \square$$

b. A sketch of the situation helps to visualize the situation and set up the algebra. Here is plot of the two curves showing their intersections, made using `SageMath`:

```
sage: plot(sqrt(x),0,5)+plot(x/2,-1,5)
```



Note that $\sqrt{x}$ is only defined for $x \geq 0$. The two curves intersect exactly when $\sqrt{x} = \frac{x}{2}$, that is, when $x = \frac{x^2}{4}$. One solution is obviously $x = 0$, but we have another when $1 = \frac{x}{4}$, *i.e.* when $x = 4$. Both curves are continuous wherever they are defined; since at $x = 1$ we have $\sqrt{1} = 1 > \frac{1}{2}$, it follows that $y = \sqrt{x}$ is above $y = \frac{x}{2}$ for $0 < x < 4$. Since the two graphs never meet again as $x \to \infty$, the finite region between the two curves is the one below $y = \sqrt{x}$ and above $\frac{x}{2}$ for $0 \leq x \leq 4$.

It follows that the area of the region is given by:

$$\begin{aligned}
\text{Area} &= \int_0^4 \left(\sqrt{x} - \frac{x}{2}\right)\, dx = \int_0^4 \left(x^{1/2} - \frac{x}{2}\right)\, dx = \left(\frac{x^{3/2}}{3/2} - \frac{1}{2}\cdot\frac{x^2}{2}\right)\Big|_0^4 \\
&= \left(\frac{2}{3}x^{3/2} - \frac{1}{4}x^2\right)\Big|_0^4 = \left(\frac{2}{3}4^{3/2} - \frac{1}{4}4^2\right) - \left(\frac{2}{3}0^{3/2} - \frac{1}{4}0^2\right) \\
&= \left(\frac{2}{3}\cdot 8 - \frac{1}{4}\cdot 16\right) - 0 = \frac{16}{3} - 4 = \frac{16}{3} - \frac{12}{3} = \frac{4}{3} \qquad \square
\end{aligned}$$

c. We will rewrite the function just a little so we can apply l'Hôpital's Rule.

$$\begin{aligned}
\lim_{x\to\infty} e^{-x}\ln(x) &= \lim_{x\to\infty} \frac{\ln(x) \to \infty}{e^x \quad \to \infty} = \lim_{x\to\infty} \frac{\frac{d}{dx}\ln(x)}{\frac{d}{dx}e^x} \\
&= \lim_{x\to\infty} \frac{1/x}{e^x} = \lim_{x\to\infty} \frac{1 \quad \to 1}{xe^x \to \infty} = 0 \qquad \square
\end{aligned}$$

d. We'll tinker algebraically with the limit the definition of the derivative hands us in this case to evaluate it. Note that if $x > 0$, then $x + h > 0$ and $\sqrt{x+h}$ is defined for all h close enough to 0.

$$\begin{aligned}
\frac{d}{dx}\sqrt{x} &= \lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h} = \lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}\cdot\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}} \\
&= \lim_{h\to 0}\frac{\left(\sqrt{x+h}\right)^2-\left(\sqrt{x}\right)^2}{h\left(\sqrt{x+h}+\sqrt{x}\right)} = \lim_{h\to 0}\frac{x+h-x}{h\left(\sqrt{x+h}+\sqrt{x}\right)} \\
&= \lim_{h\to 0}\frac{h}{h\left(\sqrt{x+h}+\sqrt{x}\right)} = \lim_{h\to 0}\frac{1}{\sqrt{x+h}+\sqrt{x}} \\
&= \frac{1}{\sqrt{x+0}+\sqrt{x}} = \frac{1}{\sqrt{x}+\sqrt{x}} = \frac{1}{2\sqrt{x}} \qquad \square
\end{aligned}$$

e. *Using slopes.* Note that $f(x) = \arctan\left(x^2\right)$ is defined and differentiable, and hence also continuous, for all x, because it is the composition of two functions which are defined and differentiable for all x.

Note that $f'(x) = \frac{d}{dx}\arctan\left(x^2\right) = \frac{1}{1+\left(x^2\right)^2}\cdot\frac{d}{dx}x^2 = \frac{2x}{1+x^4}$ is defined (and continuous and differentiable) for all x since $1+x^4 \geq 1$ is never 0. It is easy to see that $f'(x) = 0$ exactly when $x = 0$, so we have only one critical point. Since $1+x^4$ and 2 are always positive, $f'(x) < 0$ when $x < 0$ and $f'(x) > 0$ when $x > 0$. That is, $f(x)$ is decreasing for all $x < 0$ and increasing for all $x > 0$, so $x = 0$ must give both a local and absolute minimum.

It follows that the absolute minimum value of $f(x) = \arctan\left(x^2\right)$ on $(-\infty, \infty)$ is $f(0) = \arctan\left(0^2\right) = \arctan(0) = 0$. $\square$

e. *Checking critical points and "endpoints".* Again, note that $f(x) = \arctan\left(x^2\right)$ is defined and differentiable, and hence also continuous, for all x, because it is the composition of two functions which are defined and differentiable for all x. Since it is continuous for all x, $f(x)$ has no vertical asymptotes, and so we need only compare the values of $f(x)$ at any critical points with its behaviour at the ends of the interval.

$f'(x) = \frac{d}{dx}\arctan\left(x^2\right) = \frac{1}{1+\left(x^2\right)^2}\cdot\frac{d}{dx}x^2 = \frac{2x}{1+x^4}$ is defined (and continuous and differentiable) for all x since $1+x^4 \geq 1$ is never 0. It is easy to see that $f'(x) = 0$ exactly when $x = 0$, so we have only one critical point, at which we have $f(0) = \arctan\left(0^2\right) = \arctan(0) = 0$.

To check the behaviour of $f(x)$ at the ends of the interval we compute the apprpriate limits, since the interval does not actually have endpoints. Note that $\arctan(t) \to \frac{\pi}{2}$ as $t \to +\infty$.

$$\begin{aligned}
\lim_{x\to-\infty} f(x) &= \lim_{x\to-\infty}\arctan\left(x^2\right) = \frac{\pi}{2} \quad \text{since } x^2 \to +\infty \text{ as } x \to -\infty \\
\lim_{x\to-\infty} f(x) &= \lim_{x\to+\infty}\arctan\left(x^2\right) = \frac{\pi}{2} \quad \text{since } x^2 \to +\infty \text{ as } x \to +\infty
\end{aligned}$$

Since $f(0) = 0$ at the only critical point of a function which is everywhere differentiable and which approaches $\frac{\pi}{2}$ at both ends of the given interval, the value of 0 at this critical point is the absolute minimum of the function on the interval. $\square$

f. To verify that $\lim_{x\to -2} x^2 = 4$ using the ε–δ definition of limits, we need to show that if we are given any $\varepsilon > 0$, there is a $\delta > 0$ such that if $|x-(-2)| < \delta$, then $\left|x^2-4\right| < \varepsilon$.

Suppose, then, that we are given a $\varepsilon > 0$. We will try to obtain the δ we need by reverse-engineering it from the condition we are trying to get, $\left|x^2-4\right| < \varepsilon$.

$$\begin{aligned} \left|x^2-4\right| < \varepsilon &\iff |(x-2)(x+2)| < \varepsilon \iff |(x-2)\,(x-(-2))| < \varepsilon \\ &\iff |x-2|\cdot|x-(-2)| < \varepsilon \iff |x-(-2)| < \frac{\varepsilon}{|x-2|} \end{aligned}$$

Sadly, we can't use $\dfrac{\varepsilon}{|x-2|}$ as our δ without further ado because δ cannot depend on x (the major concern) and because of the small risk we could divide by 0 if x happened to equal 2. We will deal with both concerns by limiting ourselves to δs that are some convenient size less than the distance between $x=2$ (where $x-2=0$) and $x=-2$ (where we're taknig the limit). 1 is a convenient number less than $2-(-2)=4$, so we'll only accept a δ with $0 < \delta \leq 1$.

Observe that if $|x-(-2)| < 1$, then $-1 < x-(-2) < 1$, so $-3 < x < -1$, and hence $-5 < x-2 < -3$. It would follow that $3 < |x-2| < 5$, and hence that $\dfrac{1}{|x-2|} > \dfrac{1}{5}$, and thus $\dfrac{\varepsilon}{|x-2|} > \dfrac{\varepsilon}{5}$.

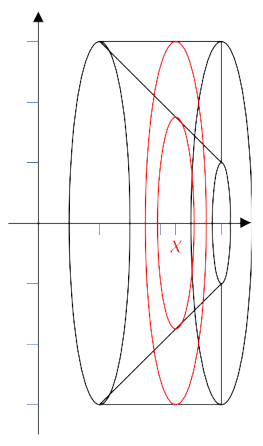
Putting the bits above together, let $\delta = \min\left(1, \frac{\varepsilon}{5}\right)$. If $|x-(-2)| < \delta$, we have $|x-(-2)| < 1$, so $\dfrac{\varepsilon}{5} < \dfrac{\varepsilon}{|x-2|}$. But then $|x-(-2)| < \delta \leq \dfrac{\varepsilon}{5} < \dfrac{\varepsilon}{|x-2|}$, so

$$\left|x^2-4\right| = |(x-2)(x+2)| = |x-2|\cdot|x-(-2)| < |x-2|\cdot\frac{\varepsilon}{|x-2|} = \varepsilon\,,$$

as required. Note again that if $|x-(-2)| < \delta \leq 1$, then $x-2 < -3$ and so $x-2 \neq 0$.

Since we can manufacture a suitable $\delta > 0$ for any $\varepsilon > 0$, $\lim_{x\to -2} x^2 = 4$. $\square$

g. *Disk/washer method.* Here is a sketch of the solid, with a typical washer cross-section drawn in:

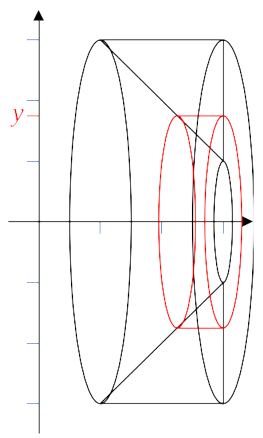


We will use the disk/washer method for computing the volume of a solid of revolution, so we will use x as the basic variable, since its axis is parallel to the axis of symmetry. The cross-section of the solid at x, for $1 \leq x \leq 3$, is a washer that has outer radius $R = 3-0 = 3$ and inner radius $r = y-0 = 4-x$, giving it an an area of $A(x) = \pi\left(R^2-r^2\right)$ $= \pi\left(3^2-(4-x)^2\right) = \pi\left(9-\left(16-8x+x^2\right)\right)$ $= \pi\left(-x^2+8x-7\right)$.

The volume of the solid can now be computed as follows:

$$\begin{aligned}\text{Volume} &= \int_1^3 A(x)\,dx = \int_1^3 \pi\left(R^2 - r^2\right)\,dx = \int_1^3 \pi\left(-x^2 + 8x - 7\right)\,dx \\ &= \pi\left(-\frac{x^3}{3} + 8\cdot\frac{x^2}{2} - 7x\right)\Big|_1^3 = \pi\left(-\frac{x^3}{3} + 4x^2 - 7x\right)\Big|_1^3 \\ &= \pi\left(-\frac{3^3}{3} + 4\cdot 3^2 - 7\cdot 3\right) - \pi\left(-\frac{1^3}{3} + 4\cdot 1^2 - 7\cdot 1\right) \\ &= \pi(-9 + 36 - 21) - \pi\left(-\frac{1}{3} + 4 - 7\right) = 6\pi - \frac{-10\pi}{3} = \frac{28\pi}{3} \qquad \square\end{aligned}$$

g. *Cylindrical shell method.* Here is a sketch of the solid, with a typical cylindrical "cross-section" drawn in:



We will use the cylindrical shell method to compute the volume of this solid of revolution, so we'll use y as the basic variable, since its axis is perpendicular to the axis of symmetry. Note that $1 \leq y \leq 3$ in the original region. The cylindrical "cross-section" at y has radius $r = y - 0 = y$ and height (length, really, since it is on its side) $h = 3 - x = 3 - (4 - y) = y - 1$, so it has an area of $A(y) = 2\pi rh = 2\pi y(y-1)$ $= 2\pi\left(y^2 - y\right)$.

The volume of the solid can now be computed as follows:

$$\begin{aligned}\text{Volume} &= \int_1^3 A(y)\,dy = \int_1^3 2\pi rh\,dy = \int_1^3 2\pi\left(y^2 - y\right)\,dy \\ &= 2\pi\left(\frac{y^3}{3} - \frac{y^2}{2}\right)\Big|_1^3 = 2\pi\left(\frac{3^3}{3} - \frac{3^2}{2}\right) - 2\pi\left(\frac{1^3}{3} - \frac{1^2}{2}\right) \\ &= 2\pi\left(9 - \frac{9}{2}\right) - 2\pi\left(\frac{1}{3} - \frac{1}{2}\right) = 2\pi\cdot\frac{9}{2} - 2\pi\cdot\frac{-1}{6} = 9\pi + \frac{\pi}{3} = \frac{28\pi}{3} \qquad \blacksquare\end{aligned}$$

4. Find the domain as well as any (and all) intercepts, vertical and horizontal asymptotes, intervals of increase, decrease and concavity, and maximum, minimum, and inflection points of $h(x) = e^{-x^2/2}$, and sketch its graph based on this information. *[14]*

SOLUTION. We will run through the given checklist:

i. Domain. $-x^2/2$ is defined for all real numbers x and e^t is defined for all real numbers t, so their composition, $h(x) = e^{-x^2/2}$, is defined for all real numbers x, too. That is, the domain of the given function is $(-\infty, \infty)$. Note that $h(x)$, as a composition of functions that are everywhere differentiable, is itself everywhere differentiable, and hence also continuous.

ii. Intercepts. $h(0) = e^{-0^2/2} = e^0 = 1$, so the y-intercept is at $y = 1$. Since $e^t > 0$ for all t, $h(x) = e^{-x^2/2} > 0$ for all x, so the function has no x-intercept.

iii. Vertical asymptotes. As noted above $h(x) = e^{-x^2/2}$ is continuous everywhere, so it cannot have any vertical asymptotes.

iv. Horizontal asymptotes. We take the limits as $x \to \pm\infty$ to check for horizontal asymptotes. Note that whether $x \to -\infty$ or $x \to +\infty$, we have $-x^2/2 \to -\infty$. Recall also that $e^t \to 0$ as $t \to -\infty$.

$$\lim_{x \to -\infty} h(x) = \lim_{x \to -\infty} e^{-x^2/2} = \lim_{t \to -\infty} e^t = 0$$
$$\lim_{x \to +\infty} h(x) = \lim_{x \to +\infty} e^{-x^2/2} = \lim_{t \to -\infty} e^t = 0$$

It follows that $h(x)$ has a horizontal asymptote of $y = 0$ in both directions. Since $h(x) = e^{-x^2/2} > 0$ for all x, it aproaches that asymptote from above in both directions.

v. Increase/decrease and max/min. As noted earlier, $h(x)$ is differentiable everywhere.

$$h'(x) = \frac{d}{dx} e^{-x^2/2} = e^{-x^2/2} \cdot \frac{d}{dx}\left(-\frac{x^2}{2}\right) = e^{-x^2}\left(-\frac{1}{2} \cdot 2x\right) = -xe^{-x^2/2}$$

Since $e^{-x^2/2} > 0$ for all x, $h'(x) = 0$ exactly when $x = 0$, $h'(x) < 0$ exactly when $x > 0$, and $h'(x) > 0$ exactly when $x < 0$. It follows that $h(x)$ is increasing on $(-\infty, 0$ and decreasing on $(0, \infty)$, and has a local and absolute maximum at $x = 0$. The maximum value occurs at the y-intercept and was previously calculated to be 1. We summarize most of this information in a table, as usual:

x	$(-\infty, 0)$	0	$(0, \infty)$
$h'(x)$	$+$	0	$-$
$h(x)$	$\uparrow$	max	$\downarrow$

vi. Concavity and inflection. It is not hard to see that $h'(x) = -xe^{-x^2/2}$ is also defined and differentiable for all x.

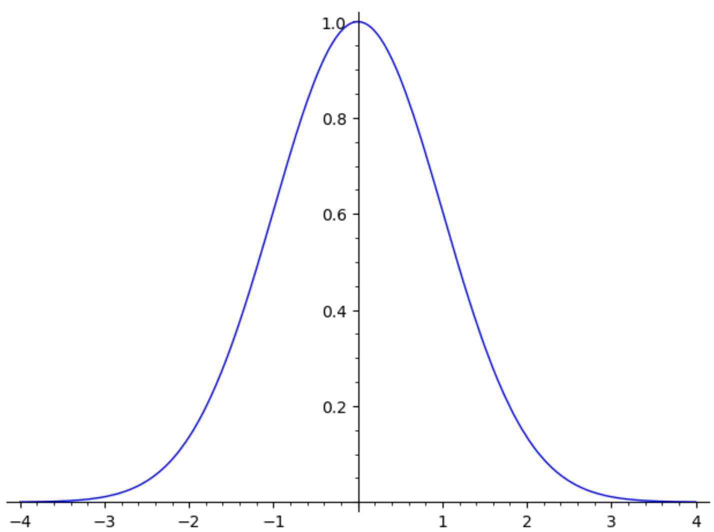
$$\begin{aligned}
h''(x) &= \frac{d}{dx}\left(-xe^{-x^2/2}\right) = \left(\frac{d}{dx}(-x)\right)e^{-x^2/2} + (-x)\frac{d}{dx}e^{-x^2/2} \\
&= (-1)e^{-x^2/2} - xe^{-x^2/2} \cdot \frac{d}{dx}\left(-\frac{x^2}{2}\right) = -e^{-x^2/2} - xe^{-x^2/2}\left(-\frac{2x}{2}\right) \\
&= -e^{-x^2/2} - xe^{-x^2/2}(-x) = -e^{-x^2/2} + x^2e^{-x^2/2} = \left(x^2 - 1\right)e^{-x^2/2} \\
&= (x+1)(x-1)e^{-x^2/2}
\end{aligned}$$

Since $e^{-x^2/2} > 0$ for all x, it follows that $h'(x) = 0$ exactly when $x = -1$ or when $x = 1$. When $x < -1$, both $x+1$ and $x-1$ are negative, so $h''(x) = (x+1)(x-1)e^{-x^2/2} > 0$; when $-1 < x < 1$, $x+1 > 0$ and $x-1 < 0$, so $h''(x) = (x+1)(x-1)e^{-x^2/2} < 0$; and when $x > 1$, both $x+1$ and $x-1$ are positive, so $h''(x) = (x+1)(x-1)e^{-x^2/2} > 0$. It follows that the graph of $h(x)$ is concave up on both $(-\infty, -1)$ and $(1, \infty)$ and concave down on $(-1, 1)$, so both $x = -1$ and $x = 1$ are inflection points of $h(x)$. As usual, we summarize most of this information in a table:

x	$(-\infty, -1)$	-1	$(-1, 1)$	1	$(1, \infty)$
$h''(x)$	$+$	0	$-$	0	$+$
$h(x)$	$\smile$	infl	$\frown$	infl	$\smile$

vii. The graph. Cheating more than a little, we get `SageMath` to do our work here:

```
sage: plot(e^(-x^2/2),-4,4)
```

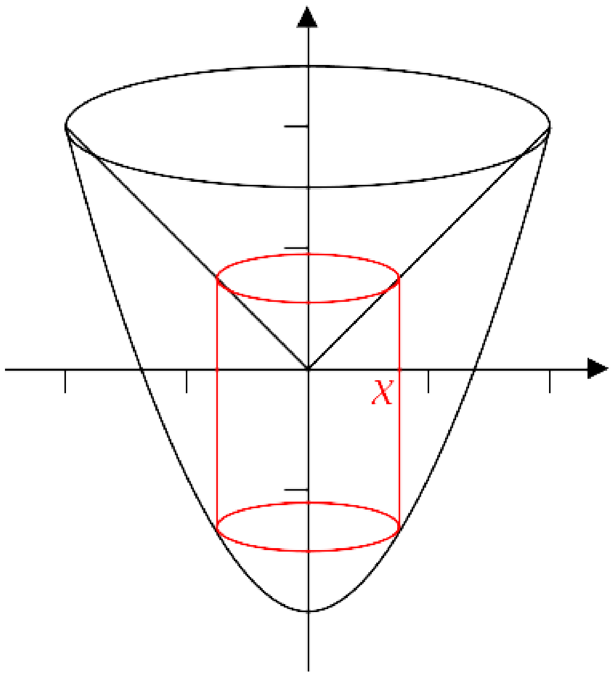


∎

Part V. Do any *two* (2) of **5–7**. *[Subtotal = 26 = 2 × 13 each]*

5. Sketch the solid obtained by revolving the region between $y = x^2 - 2$ and $y = x$, where $0 \leq x \leq 2$, about the y-axis, and find its volume.

SOLUTION. We will use the cylindrical shell method to compute the volume of the given solid of revolution. Here is sketch of the solid, with a cylindrical shell drawn in.



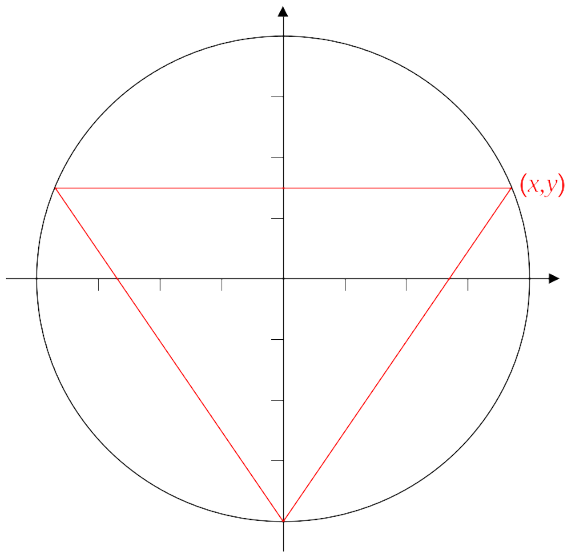
Note that since we are using cylindrical shells for our "cross-sections", we should use the variable, in this case x, whose axis is perpendicular to the axis of revolution. The shell at x, where $0 \leq x \leq 2$, has radius $r = x - 0 = x$ and height $h = x - \left(x^2 - 2\right) = -x^2 + x + 2$, and hence has area $A(x) = 2\pi rh = 2\pi x\left(-x^2 + x + 2\right) = 2\pi\left(-x^3 + x^2 + 2x\right)$.

It follows that the volume of the solid is given by:

$$\begin{aligned}
\text{Volume} &= \int_0^2 A(x)\,dx = \int_0^2 2\pi rh\,dx = \int_0^2 2\pi\left(-x^3 + x^2 + 2x\right)\,dx \\
&= 2\pi\left(-\frac{x^4}{4} + \frac{x^3}{3} + 2\cdot\frac{x^2}{2}\right)\Bigg|_0^2 = 2\pi\left(-\frac{x^4}{4} + \frac{x^3}{3} + x^2\right)\Bigg|_0^2 \\
&= 2\pi\left(-\frac{2^4}{4} + \frac{2^3}{3} + 2^2\right) - 2\pi\left(-\frac{0^4}{4} + \frac{0^3}{3} + 0^2\right) \\
&= 2\pi\left(-4 + \frac{8}{3} + 4\right) - 2\pi\cdot 0 = 2\pi\cdot\frac{8}{3} - 0 = \frac{16\pi}{3} \qquad \blacksquare
\end{aligned}$$

6. A triangle has one vertex at the point $(0,-4)$ and the other two at the points $(-x,y)$ and (x,y), where $x^2+y^2=16$. Find the maximum area of such a triangle.

SOLUTION. Here's a sketch of the setup:



Note first that we can restrict ourselves to x with $0 \le x \le 4$, because also using values of $x<0$ will simply duplicate triangles since then $-x>0$. Second, for every x with $0 \le x \le 4$, there are two possible values of y, namely $y=\pm\sqrt{16-x^2}$. Since a value of $y<0$ will give a triangle with the same base (well, top, since the triangle is point down) but smaller height than its positive counterpart, it will have smaller area. It follows that we only have to consider triangles for which the vertices (x,y) and $(-x,y)$ have $0 \le x \le 4$ and $y=\sqrt{16-x^2}$. Any such triangle will have base $b=x-(-x)=2x$ and height $h=y-(-4)=\sqrt{16-x^2}+4$, and hence have area

$$A(x)=\frac{1}{2}bh=\frac{1}{2}\cdot 2x\left(\sqrt{16-x^2}+4\right)=x\sqrt{16-x^2}+4x\,.$$

Our task is to maximize $A(x)$ on $[0,4]$; we will do so by comparing the values of $A(x)$ at the endpoints of the interval with its values at any critical point in the interval. This will work because $A(x)$ is defined and continuous on $[0,4]$ and differentiable on $(0,4)$.

First, the endpoints: $A(0)=0\sqrt{16-0^2}+4\cdot 0=0$ and $A(4)=4\sqrt{16-4^2}+4\cdot 4=4\cdot 0+16=16$.

Second, the critical points, *i.e.* the points where $A'(x)=0$, in the interval $[0,4]$.

$$\begin{aligned}A'(x)&=\frac{d}{dx}\left(x\sqrt{16-x^2}+4x\right)=\left[\frac{d}{dx}x\right]\sqrt{16-x^2}+x\frac{d}{dx}\sqrt{16-x^2}+4\\&=1\sqrt{16-x^2}+x\frac{1}{2\sqrt{16-x^2}}\cdot\frac{d}{dx}\left(16-x^2\right)+4\\&=\sqrt{16-x^2}+\frac{x}{\sqrt{16-x^2}}\cdot(-2x)+4=\sqrt{16-x^2}-\frac{x^2}{\sqrt{16-x^2}}+4\end{aligned}$$

Now to find the critical points:

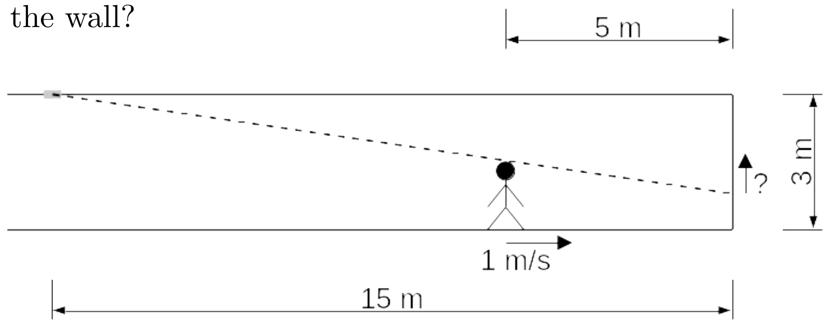
$$\begin{aligned}
A'(x) = 0 \implies & \sqrt{16-x^2} - \frac{x^2}{\sqrt{16-x^2}} + 4 = 0 \\
\implies & \left(\sqrt{16-x^2}\right)^2 - \frac{x^2}{\sqrt{16-x^2}}\sqrt{16-x^2} + 4\sqrt{16-x^2} = 0\sqrt{16-x^2} \\
\implies & 16 - x^2 - x^2 + 4\sqrt{16-x^2} = 0 \\
\implies & 4\sqrt{16-x^2} = 2x^2 - 16 \\
\implies & 2\sqrt{16-x^2} = x^2 - 8 \\
\implies & 4\left(16-x^2\right) = \left(2\sqrt{16-x^2}\right)^2 = \left(x^2-8\right)^2 \\
\implies & 64 - 4x^2 = x^4 - 16x^2 + 64 \\
\implies & x^4 - 12x^2 = 0 \\
\implies & x^2\left(x^2 - 12\right) = 0 \\
\implies & x = 0 \text{ or } x = \pm\sqrt{12} = \pm 2\sqrt{3}
\end{aligned}$$

$x = 0$ is an endpoint of the interval $[0, 4]$ and $A(0) = 0$ has already been determined. $x = -2\sqrt{3}$ is not in the interval, so it need not be considered. $x = +2\sqrt{3}$ is in the interval and

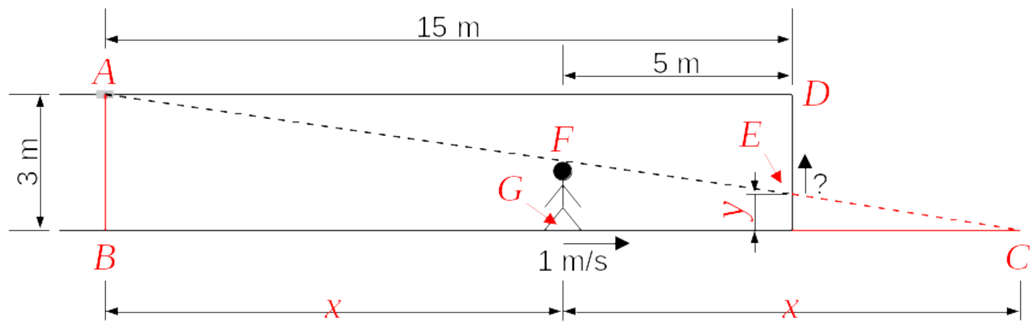
$$\begin{aligned}
A\left(2\sqrt{3}\right) &= 2\sqrt{3}\cdot\sqrt{16-\left(2\sqrt{3}\right)^2} + 4\cdot 2\sqrt{3} = 2\sqrt{3}\cdot\sqrt{16-12} + 8\sqrt{3} \\
&= 2\sqrt{3}\cdot\sqrt{4} + 8\sqrt{3} = 2\sqrt{3}\cdot 2 + 8\sqrt{3} = 4\sqrt{3} + 8\sqrt{3} = 12\sqrt{3} \approx 20.7846\,.
\end{aligned}$$

Of our three candidates for the maximum possible value of $A(x)$ on the interval $[0, 4]$, $A\left(2\sqrt{3}\right) = 12\sqrt{3} \approx 20.7846$ is clearly the largest, so it is the maximum possible area of a triangle with one vertex at the point $(0, -4)$ and the other two at the points $(-x, y)$ and (x, y), where $x^2 + y^2 = 16$. ■

7. A straight and narrow corridor has a ceiling 3 m above the floor. The only illumination is from a ceiling panel 15 m from where the corridor ends in a vertical wall. Stick Person, who is 1.5 m tall, walks past the lamp towards the blank wall at 1 m/s. How quickly is the top of the shadow Stick casts on the wall rising at the instant that Stick is 5 m from the wall?



SOLUTION. *One way.* We'll use some geometry to set up the calculus. Consider the following modified version of the diagram provided.



As in the modified diagram, let A be the location of the ceiling panel, B be the point on the floor directly underneath the ceiling panel, C be the point where the tip of Stick Person's shadow would be on the floor if the corridor did not end at the wall, D be the point where the ceiling meets the wall, E be the location on the wall of the top of the shadow cast by Stick, F be the top of Stick's head, and G be the point on the floor directly below Stick.. Also, let x be the horizontal distance of Stick from the ceiling panel (*i.e.* the distance between B and G) and let y be the height of the shadow that Stick casts on the wall.

Observe that $\triangle ABC$ is similar to (*i.e.* has the same proportions as) $\triangle FGC$, since they share a common angle at C and each has a right angle, at B and G, respectively. It follows that corresponding sides of these two triangles are always in the same proportion; since AB is 3 m long and FG, being Stick Person's height, is 1.5 m long, BC is also twice as long as GC. This means that BG, which has length x, is just as long as GC, which therefore also has length x. In turn, this means that BC has length $2x$.

Since AD is parallel to BC and AC is the straight line line from A to C, the narrow angle at A, $\angle DAC$, is equal to the narrow angle at C, $\angle BCA$. Since we have right angles at D and B, it follows that $\triangle EDA$ is also similar to $\triangle ABC$, so corresponding sides of these two triangles are also always in the same proportion. In particular, this means that

ED is in the same proportion to AB as DA is to BC, that is, that $\frac{3-y}{3} = \frac{15}{2x}$. This gives us the basic relationship between x and y.

We rearrange the equation obtained above to solve for y:

$$\frac{3-y}{3} = \frac{15}{2x} \implies 1 - \frac{y}{3} = \frac{15}{2x} \implies \frac{y}{3} = 1 - \frac{15}{x} \implies y = 3 - \frac{45}{2x}$$

Taking derivatives:

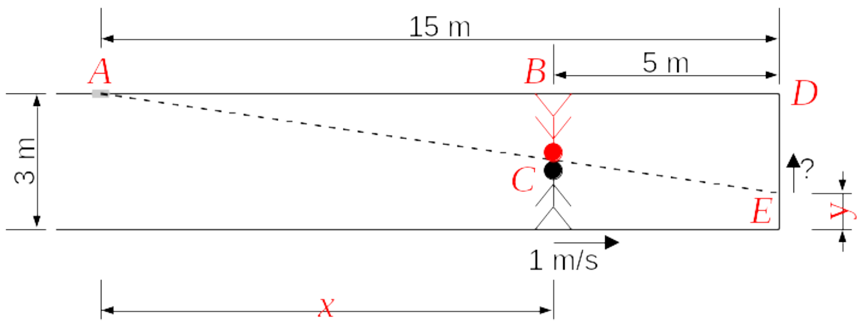
$$\frac{dy}{dt} = \frac{d}{dt}\left(3 - \frac{45}{2x}\right) = 0 - \left[\frac{d}{dx}\left(\frac{45}{2x}\right)\right] \cdot \frac{dx}{dt} = -\frac{-45}{2x^2} \cdot \frac{dx}{dt} = \frac{45}{2x^2} \cdot \frac{dx}{dt}$$

We want to know $\frac{dy}{dt}$ at the instant that Stick Person is 5 m from the wall, that is, when $x = 15 - 5 = 10$ m. We are given that $\frac{dx}{dt} = 1$ m/s. Putting all this together tells us that the the top of the shadow Stick casts on the wall rising is rising at a rate of

$$\left.\frac{dy}{dt}\right|_{x=10} = \frac{45}{2 \cdot 10^2} \cdot 1 = \frac{45}{200} = \frac{9}{40} = 0.225 \ m/s$$

at the given instant. (Whew!) *square*

SOLUTION. *A simpler way.* We'll use a little less geometry to set up the calculus. Consider the following modified version of the diagram provided.



Imagine we have Stick Person's adhesive cousin, Sticky Individual, who is also 1.5 m tall, walking on the ceiling directly above Stick. This time, let B be Sticky's position on the ceiling, C be the top of Sticky's head (and Stick's too :-), D be the point where the ceiling meets the wall at the end of the corridor, and E be the top of Stick's shadow on that wall, *i.e.* where the line passing through A and C meets the wall. let x be the horizontal distance of Sticky (and Stick) from the ceiling panel (*i.e.* the distance between A and B) and let y be the height of the shadow that Stick casts on the wall. Note that the distance between D and E is $3 - y$.

Since they share a common angle at A and each have a right angle elsewhere (at B and D), $\triangle ABC$ and $\triangle ADE$ are similar. Since similar triangles must have corresponding sides in the same proportion, it follows that $\frac{3-y}{15} = \frac{1.5}{x}$, so $3 - y = \frac{22.5}{x}$, and hence $y = 3 - \frac{22.5}{x}$. Differentiating gives:

$$\frac{dy}{dx} = \frac{d}{dx}\left(3 - \frac{22.5}{x}\right) = -22.5\left(-\frac{1}{x^2}\right) = \frac{22.5}{x^2}$$

Since $\frac{dx}{dt} = 1\ m/s$, at the instant that $x = 15 - 5 = 10\ m$, we have:

$$\left.\frac{dy}{dt}\right|_{x=10} = \frac{22.5}{10^2} \cdot 1 = \frac{22.5}{100} = 0.225\ m/s \qquad \blacksquare$$

[Total = 100]

Part W. *Bonus problems!* If you feel like it, do one or both of these.

2^3. Find a formula in terms of n for the sum $\sum_{i=1}^{n} i(i+4) = 1 \cdot 5 + 2 \cdot 6 + \cdots + n(n+4)$. *[1]*

3^2. Write a haiku touching on calculus or mathematics in general. *[1]*

What is a haiku?

seventeen in three:
five and seven and five of
syllables in lines

We hope that you enjoyed the course. Enjoy the break!