Just for fun! Not on quiz or exam!

Why the standard normal density is a valid density,

or, Behold the power of fully operational multivariate calculus! (With apologies to former Senator Palpatine...)

A function $f(x)$ is a valid probability density function if

(1) $f(x)$ is defined and integrable on $\mathbb{R}$,

(2) $f(x) \geq 0$ for all $x \in \mathbb{R}$,

and (3) $\int_{-\infty}^{\infty} f(x)\,dx = 1$.

The standard normal density is

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}.$$

Conditions (1) & (2) are easy to check for $\varphi(x)$. (3) will take a bit of work...

<u>To Show:</u> $\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,dx = 1$

We'll actually show (and it's good enough) that $\left(\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx \right)^2 = 1.$

$$\left[\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx \right]^2$$

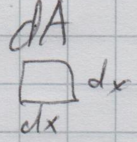
$$= \left[\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx \right] \left[\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx \right]$$

$$= \left[\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx \right] \left[\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy \right]$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2} \left[\boxed{\frac{1}{\sqrt{2\pi}}} \int_{-\infty}^{\infty} e^{-y^2/2} dy \right] dy$$

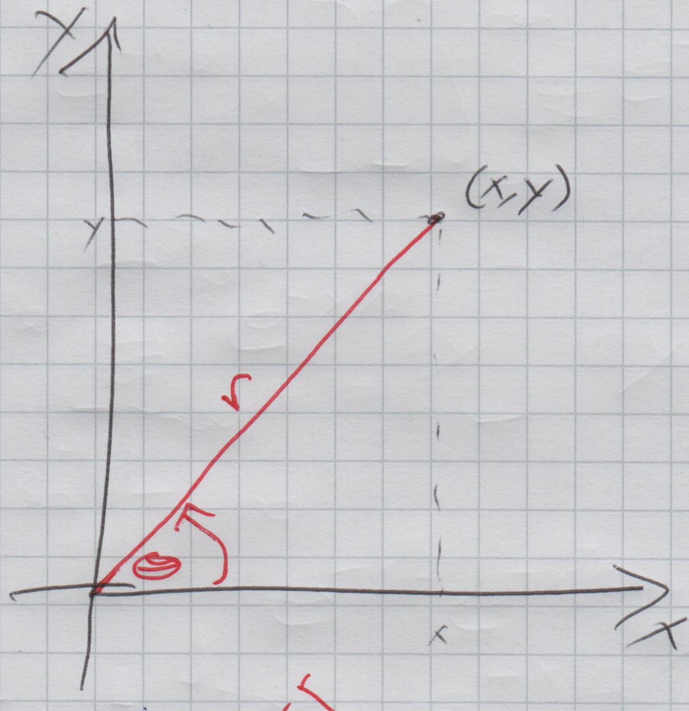
$$= \left(\frac{1}{\sqrt{2\pi}} \right)^2 \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} e^{-x^2/2} e^{-y^2/2} dy \right] dy$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)/2} \, dy \, dx$$



$$= \frac{1}{2\pi} \iint_{\mathbb{R}^2} e^{-(x^2+y^2)/2} \, dA$$

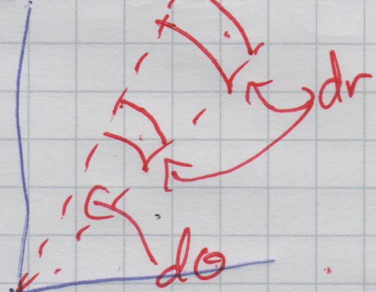
We're going to switch this to (& evaluate in) polar coordinates:



$r = \sqrt{x^2+y^2}$

$x = r\cos(\theta)$

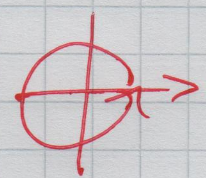
$y = r\sin(\theta)$



$dA = r\,dr\,d\theta$

$$= \frac{1}{2\pi} \iint_{\mathbb{R}^2} e^{-(x^2+y^2)/2}\, dA$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \int_0^{\infty} e^{-r^2/2}\, r\, dr\, d\theta$$



$0 \leq \theta \leq 2\pi$ radians

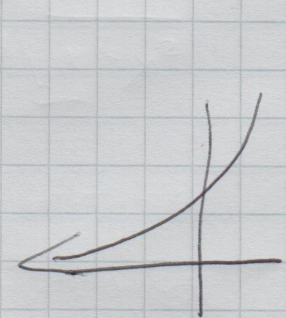
For the inner integral, we'll use the substitution $u = -r^2/2$

$du = \frac{-\cancel{2}r}{\cancel{2}}\, dr$

$(-1)\,du = r\,dr$

r	u	
0	0	$= -0^2/2$
∞	$-\infty$	$= -\infty^2/2$

$$= \frac{1}{2\pi} \int_0^{2\pi} \int_0^{-\infty} e^{u} (-1)\, du\, d\theta$$



$$= \frac{1}{2\pi} \int_0^{2\pi} \int_{-\infty}^{0} e^{u}\, du\, d\theta$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \left(e^{u} \Big|_{-\infty}^{0} \right) d\theta = \frac{1}{2\pi} \int_0^{2\pi} \left(\overset{1}{\overset{\shortparallel}{e^{0}}} - \overset{0}{\overset{\shortparallel}{e^{-\infty}}} \right) d\theta$$

$$= \frac{1}{2\pi} \int_0^{2\pi} 1\, d\theta = \frac{1}{2\pi}\, \theta \Big|_0^{2\pi} = \frac{1}{2\pi} \cdot 2\pi - \cancel{\frac{1}{2\pi} \cdot 0} = 1$$

✓