The Fundamental Theorem of Calculus

or, what integrals have to do with derivatives.

The Fundamental Theorem of Calculus

(I) Suppose $F'(x) = f(x)$ on $[a,b]$, where $f(x)$ is defined and integrable on $[a,b]$. Then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

(II) Suppose $f(x)$ ~~[illegible]~~ is defined and integrable on $[a,b]$. If we define $F(x) = \int_a^x f(x)\,dx$ (for $a \le x \le b$),

then $F'(x) = f(x)$.

This reduces the problem of computing $\int_a^b \underset{\text{"integrand"}}{f(x)}\,dx$ to finding an "anti-derivative" of $f(x)$, ie a function $F(x)$ s.t. $F'(x) = f(x)$, and then evaluating at the endpoints.

There are two practical problems here:

1) Not every function $f(x)$ has a "closed form" anti-derivative.

eg $f(x) = e^{-x^2}$ has no antiderivative that you can express in a finite way using familiar functions and operations (ie no "closed form").

[And what it's for in statistics is to find the areas under it...]

[This function matters, since it is a scaled version of $\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$, which is the classic Gaussian distribution (a.k.a. standard normal distribution, a.k.a. "the bell curve"), which is critically important in statistics.]

2) Even for functions that do have "closed form" anti-derivatives, finding them cannot be reduced to a mechanical procedure to the degree that one can for computing derivatives. In practice, we have a variety of techniques we can apply in many cases (but not all!) and we rely on pattern-recognition to use

So we'll be looking to develop

(1) a library of anti-derivatives of various common functions [run various derivatives we know in reverse]

& (2) a suite of techniques for computing & manipulating anti-derivatives in various situations.

Terminology & notation:

"$\int f(x)\,dx$" $\equiv$ "the indefinite integral of $f(x)$"

denotes a generic anti-derivative of $f(x)$.

Note that if $F'(x) = f(x)$, then

$G(x) = F(x) + C$ for any constant C

is also a function such that $G'(x) = f(x)$.

Here are some common anti-derivatives:

1) $\int 1\,dx = x + C$ because $\frac{d}{dx}(x+C) = 1 + 0 = 1$.

2) $\int x\,dx = \frac{x^2}{2} + C$ because $\frac{d}{dx}\left(\frac{x^2}{2} + C\right) = \frac{2x}{2} + 0 = x$

3) (Power Rule for Integration) [also on the list of integration techniques]

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$$

(for all real numbers $n \neq -1$)

since $\frac{d}{dx}\left(\frac{x^{n+1}}{n+1} + C\right)$

$= \frac{(n+1)x^n}{n+1} + 0 = x^n$

and $\int x^{-1}\,dx = \int \frac{1}{x}\,dx = \ln(x) + C$ since $\frac{d}{dx}(\ln(x) + C)$

$= \frac{1}{x} + 0 = \frac{1}{x}$

4) $\int \cos(x)\,dx = \sin(x) + C$ since $\frac{d}{dx}(\sin(x) + C) = \cos(x) + 0 = \cos(x)$

$\int \sin(x)\,dx = -\cos(x) + C$ since $\frac{d}{dx}(-\cos(x) + C) = -(-\sin(x)) + 0$

$= \sin(x)$

5) $\int e^x dx = e^x + C$ since $\frac{d}{dx}(e^x + C) = e^x + 0 = e^x$

6) $\int \frac{1}{x^2+1} dx = \arctan(x) + C$ since $\frac{d}{dx}(\arctan(x) + C)$
$= \frac{1}{1+x^2} + 0 = \frac{1}{1+x^2}$

... and so on. We'll add to this as we develop techniques that let us work out things like $\int \tan(x)\,dx = \int \frac{\sin(x)}{\cos(x)} dx$.

For our suite of techniques, we can draw on the properties of the definite integral for inspiration:

1° $\int (f(x) + g(x))\,dx = \int f(x)\,dx + \int g(x)\,dx$ (Sum Rule for integrals)

2° $\int c f(x)\,dx = c \int f(x)\,dx$ (Multiplication by constants)

3° $\int x^n dx = \left\{ \begin{array}{ll} \frac{x^{n+1}}{n+1} & \text{if } n \neq -1 \\ \ln(x) & \text{if } n = -1 \end{array} \right\} + C$ (Power Rule for integrals)

4° (Substitution Rule)

This is the counterpart of the Chain Rule for derivatives.

Chain Rule: $\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x)$

Substitution Rule: $\int f'(g(x)) \cdot g'(x)\, dx = f(g(x)) + C$

This relies on picking apart the integrand into the components $f'(g(x))$ and $g'(x)$, which is not always obvious [when possible].

Es $\int \tan(x)\, dx = \int \frac{\sin(x)}{\cos(x)}\, dx$

Which accessible (you can factor it out) part is a derivative of something else. Here $\sin(x)$ is a accessible and is the negative of the derivative of $\cos(x)$ ...

Substitute $u = \cos(x)$

so $\frac{du}{dx} = -\sin(x)$

so $du = [-\sin(x)]\, dx$

so $(-1)\, du = \sin(x)\, dx$

$$\text{Thus } \int \tan(x)\,dx = \int \frac{\sin(x)}{\cos(x)}\,dx = \int \underbrace{\frac{1}{\cos(x)}}_{f(g(x))} \cdot \underbrace{\sin(x)}_{g'(x)}\,dx$$

$$= \int \frac{1}{u}(-1)\,du = -\int \frac{1}{u}\,du$$

Note that we're using substitution to simplify the integrand.

Indefinite integrals always should return to the original variable.

$$= -\int u^{-1}\,du = -\ln(u) + C$$

[by the Power Rule]

put back in terms of x

$$= -\ln(\cos(x)) + C$$

We can rewrite this in many ways, using facts about trig functions and logarithms.

eg

$$= (-1)\ln(\cos(x)) + C$$

$$= \ln\left([\cos(x)]^{-1}\right) + C$$

$$= \ln\left(\frac{1}{\cos(x)}\right) + C$$

$$= \ln(\sec(x)) + C.$$

Next time: beat up on the Substitution Rule...