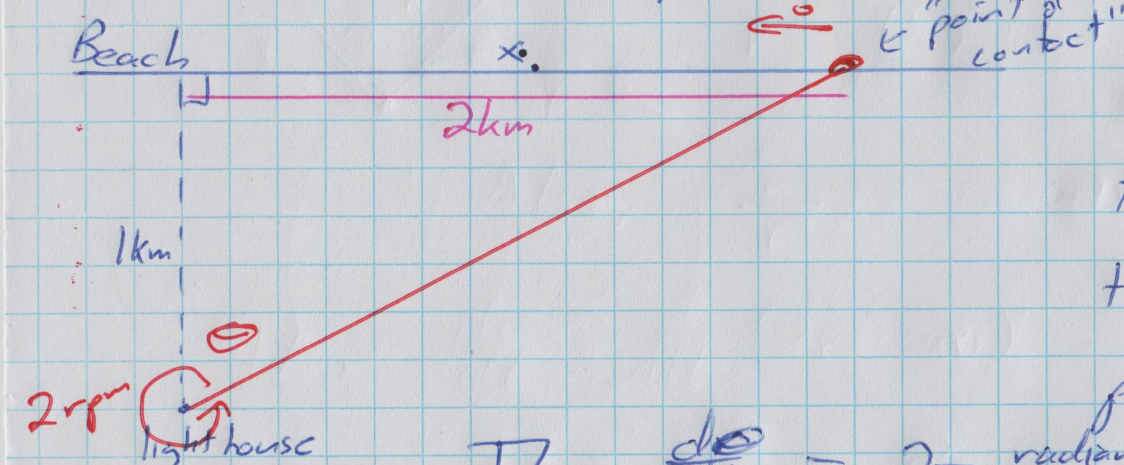
Related Rates III - Twice more, with angles.

1° A lighthouse is 1km offshore from a straight beach. The light rotates at 2 revolutions per minute, counterclockwise, casting a straight beam. At the instant that the point where the beam meets the shore is approaching the nearest point on the beach to the lighthouse, but is still 2km away, how fast is the point of contact moving along the beach?



Let θ be the angle between the beam and the line joining the lighthouse to the nearest point on the beach.

Then $\frac{d\theta}{dt} = 2\pi \frac{\text{radians}}{\text{revolution}} \cdot 2 \frac{\text{revolutions}}{\text{minute}} = 4\pi\ \text{rad}/\text{min}$,

and the distance x between the point of contact and the nearest point on the beach to the lighthouse is related to θ by

$$x = \frac{\text{opposite}}{\text{adjacent}} = \frac{x}{1} = \tan(\theta).$$

We want to know $\frac{dx}{dt}$:

$$x = \tan(\theta) \implies \frac{dx}{dt} = \frac{d}{dt}\tan(\theta) = \left(\frac{d}{d\theta}\tan(\theta)\right)\cdot\frac{d\theta}{dt}$$

$$= \sec^2(\theta)\cdot 4\pi$$

When $x=2$, $\left.\frac{dx}{dt}\right|_{x=2} = \sec^2(\theta)\cdot 4\pi$ for the θ you have at $x=2$

$$= (1+x^2)\cdot 4\pi\,\Big|_{x=2}$$

$$= (1+2^2)\cdot 4\pi = 5\cdot 4\pi$$

$$= 20\pi \text{ km/min}$$

$$\left[\begin{aligned}\sec^2(\theta) &= 1+\tan^2(\theta)\\ &= 1+x^2\end{aligned}\right]$$

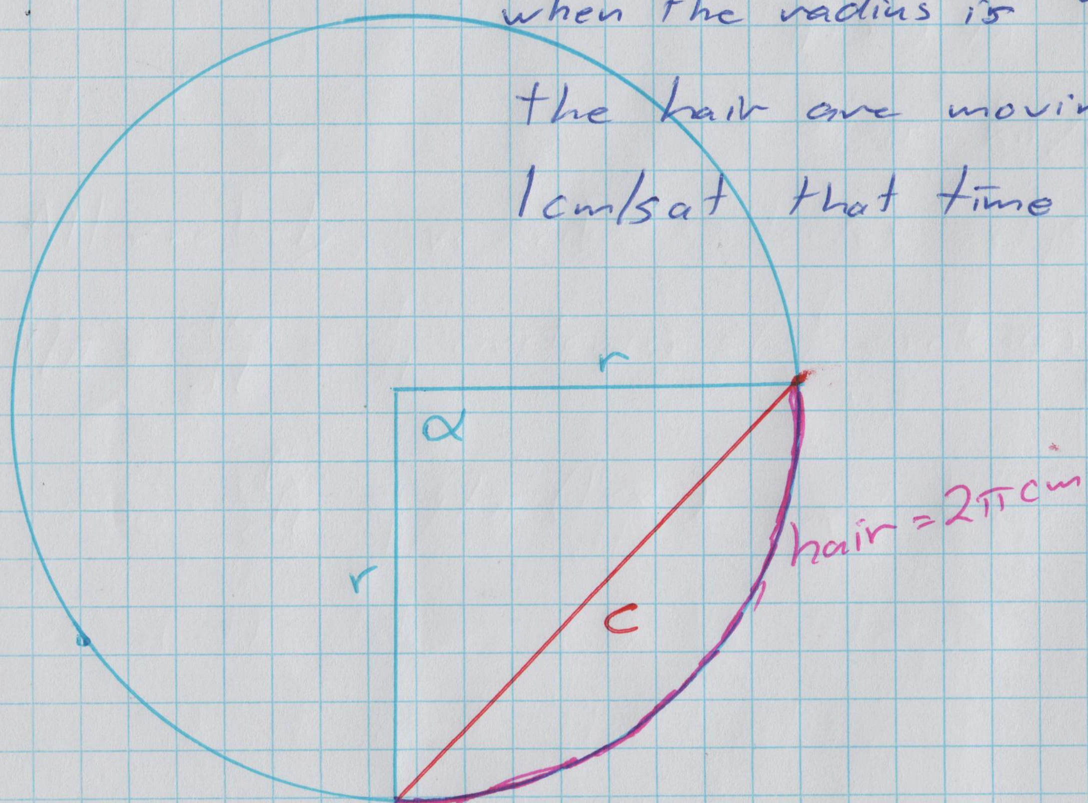
Alternatively, rewrite $x=\tan(\theta)$ as $\theta = \arctan(x)$. Then

$$4\pi = \frac{d\theta}{dt} = \frac{d}{dt}\arctan(x) = \left(\frac{d}{dx}\arctan(x)\right)\cdot\frac{dx}{dt} = \frac{1}{1+x^2}\cdot\frac{dx}{dt}$$

So $\frac{dx}{dt} = 4\pi(1+x^2)$, and hence $\left.\frac{dx}{dt}\right|_{x=2} = 4\pi(1+2^2) = 20\pi$.

2° [Taken from Assignment #3 in MA 110 in 1997-1998.]

A hair 2π cm long lies as straight as possible on a spherical balloon while that is being inflated. [The balloon is spherical at all times, the hair stretch or shrink, and remains as straight as possible while remaining in contact with the balloon.] How is the radius of the balloon changing when the radius is 4 cm, if the ends of the hair are moving apart at a rate of 1 cm/s at that time?



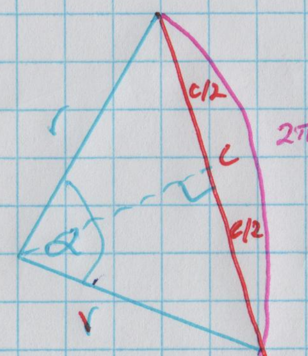
Cross-section of balloon

At the instant that $r = 4$, how is the length of the chord c changing? (Assuming that $\frac{dc}{dt} = 1$ cm/s at that instant.)

Let α be the angle made by the radii from the centre of the circle to the ends of the hair.

<u>Fact</u>: The arc-length subtended by a central angle of α in a circle of radius r is given by $r\alpha$, if α is measured in radians.

$$\left(\frac{\alpha}{2\pi}\cdot \underset{2\pi r}{\underset{\shortparallel}{\text{circumference}}} = \text{arclength}\right)$$



$$\sin\left(\frac{\alpha}{2}\right) = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{c/2}{r} = \frac{c}{2r}$$

so $c = 2r\sin\left(\frac{\alpha}{2}\right)$.

But $2\pi = r\alpha$, so

$$\alpha = \frac{2\pi}{r}.$$

$$\therefore\ 1_{cm/s} = \frac{dc}{dt} = \frac{d}{dt}\left(2r\sin\left(\frac{\alpha}{2}\right)\right)$$

$$= \frac{d}{dt}\left(2r\sin\left(\frac{\cancel{2}\pi}{\cancel{2}r}\right)\right).$$

$$= \frac{d}{dr}\left(2r\sin\left(\frac{\pi}{r}\right)\right)\cdot\frac{dr}{dt} = \left[2\sin\left(\frac{\pi}{r}\right) + 2r\cos\left(\frac{\pi}{r}\right)\cdot\frac{d}{dr}\left(\frac{\pi}{r}\right)\right]\cdot\frac{dr}{dt}$$

$$= 2\left[\sin\left(\frac{\pi}{r}\right) + \cancel{r}\cos\left(\frac{\pi}{r}\right)\cdot\left(\frac{-\pi}{r^{\cancel{2}}}\right)\right]\frac{dr}{dt} = 2\left[\sin\left(\frac{\pi}{r}\right) - \frac{\pi}{r}\cos\left(\frac{\pi}{r}\right)\right]\frac{dr}{dt}$$

$$\therefore \frac{dr}{dt} = \frac{1}{2\left[\sin\left(\frac{\pi}{r}\right) - \frac{\pi}{r}\cos\left(\frac{\pi}{r}\right)\right]}$$

and when $r=4$, $$\left.\frac{dr}{dt}\right|_{r=4} = \frac{1}{2\left[\sin\left(\frac{\pi}{4}\right) - \frac{\pi}{4}\cos\left(\frac{\pi}{4}\right)\right]}$$

$$\left[\sin\left(\frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}\right]$$

$$= \frac{1}{2\left[\frac{1}{\sqrt{2}} - \frac{\pi}{4}\cdot\frac{1}{\sqrt{2}}\right]}$$

$$\left[\frac{2}{\sqrt{2}} = \sqrt{2}\right]$$

$$= \frac{1}{\sqrt{2} - \frac{\pi}{4}\sqrt{2}}$$

$$= \frac{1}{\sqrt{2}\left(1-\frac{\pi}{4}\right)} = \frac{4}{(4-\pi)\sqrt{2}} = \frac{2\sqrt{2}}{4-\pi} \approx 3.3$$

$\therefore$ At the instant in question, $\frac{dr}{dt} \approx 3.3$ cm/s

$$\frac{2\sqrt{2}}{4-\pi} \text{ cm}$$