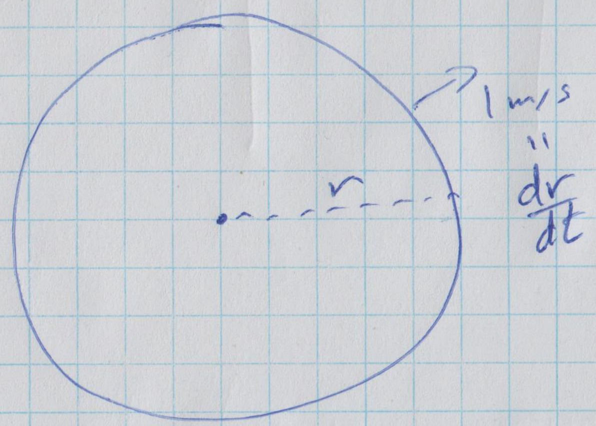
Related Rates

Problems in which one is given one rate of change and is asked to compute another.

(1) Drop a pebble into a still pond, creating a circular ripple that spreads outward at a constant rate of 1m/s. How are the length of the perimeter and the area enclosed by the ripple changing when after two seconds?



Given that $\frac{dr}{dt} = 1$, compute $\frac{dP}{dt}$ & $\frac{dA}{dt}$ (at the instant two seconds after impact). Note $r = 2m$ after $2s$ (since $r=0$ at $t=0$ and $\frac{dr}{dt} = 1$).

P = Perimeter = circumference $= 2\pi r$

A = Area $= \pi r^2$

$$\frac{dP}{dt} = \frac{d}{dt}(2\pi r) = 2\pi \frac{dr}{dt} = 2\pi \cdot 1 = 2\pi \ \text{m/s}$$

(Note that the particular doesn't matter.)

$$\frac{dA}{dt} = \frac{d}{dt}\pi r^2 = \left(\frac{d}{dr}\pi r^2\right)\cdot\frac{dr}{dt} = 2\pi r \cdot \frac{dr}{dt} = 2\pi r \cdot 1 = 2\pi r \ \text{m}^2/\text{s}$$

Chain Rule

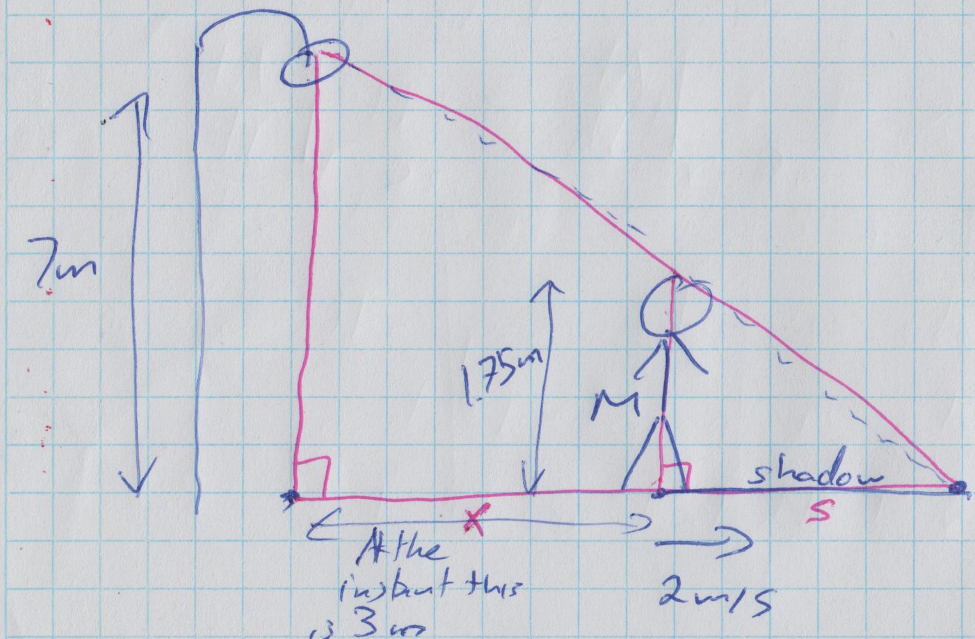
Note that r, and hence the time, matter here.

$$\left.\frac{dA}{dt}\right|_{t=2s.} = \left.\frac{dA}{dt}\right|_{r=2m} = 2\pi \cdot 2 = 4\pi \ \text{m}^2/\text{s}$$

$\therefore$ The perimeter is changing at a rate of 2π m/s and the area is changing at a rate of 4π m^2/s at the instant that $r = 2$m (ie after 2 s).

2 M is a stick person (because that's all I can draw) who walks directly beneath a street light that is 7m above the ground and continues walking away from it (on flat pavement) at a constant rate of 2m/s.
M is 1.75 m tall.

a) How is the tip of M's shadow moving when M is 3m past the light?



b) How is the length of the shadow ~~moving~~ changing when M is 3m past the light.

Let x be the distance from the base of the light to M & let s be the length of M's shadow.

We know that $\frac{dx}{dt} = 2$ m/s.

The triangle formed by the light and the tip of the shadow, and the triangle formed by M and the tip of the shadow, are <u>similar</u>.

This means that the triangle have all of their corresponding sides in the same proportions, so $\frac{1.75}{7} = \frac{s}{x+s}$

ie $\frac{s}{x+s} = \frac{1.75}{7} = \frac{7/4}{7} = \frac{1}{4} \iff s = \frac{1}{4}(x+s) = \frac{1}{4}x + \frac{1}{4}s$

$\iff \frac{3}{4}s = \frac{1}{4}x \iff x = 3s$

b) The length of the shadow is changing at a rate of $\frac{ds}{dt}$, and $\frac{ds}{dt} = \frac{d}{dt}\left(\frac{x}{3}\right) = \frac{1}{3}\cdot\frac{dx}{dt} = \frac{1}{3}\cdot 2 = \frac{2}{3}$ m/s.

a) The tip of the shadow is moving along the ground at the rate of change of $x+s$, ie $\frac{d(x+s)}{dt}$.

$$\frac{d(x+s)}{dt} = \frac{dx}{dt} + \frac{ds}{dt} = 2 + \frac{2}{3} = \frac{6}{3} + \frac{2}{3} = \frac{8}{3} \text{ m/s}.$$

Note that both of these are <u>not</u> dependent on how ~~far~~ M is from the light.