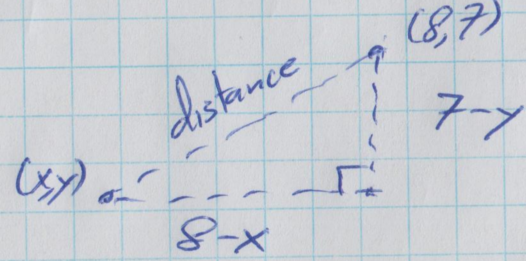
Optimization II - a distance problem the calculus way

Problem: Find the least distance from the point $(8,7)$ to a point on the line $y = 7x+1$.

The distance from a point (x,y) to $(8,7)$ is given by the formula for the length of the hypotenuse of a right triangle.



$$+\sqrt{(7-y)^2+(8-x)^2}$$

$$= +\sqrt{(y-7)^2+(x-8)^2}.$$

If $y = 7x+1$, this gives the formula

$$\sqrt{(7x+1-7)^2+(x-8)^2} = \sqrt{(7x-6)^2+(x-8)^2}$$

$$= \sqrt{49x^2-84x+36+x^2-16x+64} = \sqrt{50x^2-100x+100}$$

$$= 5\sqrt{2x^2-4x+4}$$

So we need to find the minimum value of

$$f(x) = 5\sqrt{2x^2-4x+4} \quad \text{on} \quad (-\infty, \infty)$$

(It's easy to see that $\lim_{x\to\pm\infty} f(x) = +\infty$. Check yourself!)

Find the critical points: - points where $f'(x) = 0$

- points where $f'(x)$ is undefined

$$f'(x) = \frac{d}{dx} 5\sqrt{2x^2-4x+4} = 5 \cdot \frac{1}{2\sqrt{2x^2-4x+4}} \cdot (4x-4)$$

$= 0$ when $x=1$ since that's when the numerator is 0

is undefined when $2x^2-4x+4 \leq 0$

Is $2x^2-4x+4 = 0$ ever?

$$\Leftrightarrow x^2-2x+2=0$$

$$\Leftrightarrow x = \frac{-(-2) \pm \sqrt{(-2)^2-4\cdot1\cdot2}}{2\cdot1} = \frac{2\pm\sqrt{4-8}}{2} = \frac{2\pm\sqrt{-4}}{2}$$

so there are no real solutions. Hence $f'(x)$ is never undefined.

So the minimum occurs when $x=1$,

~~ie~~ it is $f(1) = 5\sqrt{2\cdot 1^2 - 4\cdot 1 + 4} = 5\sqrt{2}$
$\approx 7. \ldots$

So the point closest to $(8,7)$ on the line $y=7x+1$
is the point $(1, 7\cdot 1+1) = (1,8)$.

How we could we have made it easier?
One way is to minimize the square of distance instead of distance, work with $g(x) = (y-7)^2+(x-8)^2$
$= 50x^2-100x+100.$

Then $g'(x) = 100x-100$ (defined everywhere!)
$= 100(x-1) = 0 \iff x=1.$

The minimum distance $= \sqrt{g(x)} = \sqrt{50\cdot 1^2-100+100} = \sqrt{50} = 7.\ldots$.

This avoids the problems with definability and has a simpler derivative.

Next time: related rates.