Curve Sketching III

Let's apply the process to $f(x) = \frac{x^2+3x+1}{x^2+2x+1}$ on $(-\infty,\infty)$

We'll add a new step at the beginning:

-1° Simplify the expression of the function (if you can).

$$f(x) = \frac{x^2+3x+1}{x^2+2x+1} = \frac{x^2+2x+1}{x^2+2x+1} + \frac{x}{x^2+2x+1} = 1 + \frac{x}{x^2+2x+1}$$

$$= 1 + \frac{x}{(x+1)^2} = 1+x(x+1)^{-2}$$

0° Intercepts: y-int. (plug in $x=0$) $= 1 + \frac{0}{(0+1)^2} = 1+0 = \boxed{1}$ ✓

x-int (plug in $y=0$) $0 = 1 + \frac{x}{(1+x)^2}$

$\Rightarrow 1+(x+1)^2 = 0$

$\Rightarrow x^2+3x+1 = 0$

$\Rightarrow x = \frac{-3 \pm \sqrt{3^2-4\cdot1\cdot1}}{2\cdot1}$

$= \frac{-3\pm\sqrt{5}}{2}$

$= \boxed{\frac{-3-\sqrt{5}}{2}}$ or $\boxed{\frac{-3+\sqrt{5}}{2}}$

1° Vertical Asymptotes: The def'n of $f(x) = 1 + \frac{x}{(1+x)^2}$ is defined & continuous & differentiable everywhere except when $1+x=0$ ie when $x=-1$.

$$\lim_{x\to -1^-} \Big(\underset{\downarrow \atop 1}{1} + \frac{\overset{\nearrow -1}{x}}{\underbrace{(1+x)^2}_{\to 0^+}}\Big) = -\infty \qquad \lim_{x\to -1^+} \Big(1 + \frac{\overset{\to -1}{x}}{\underbrace{(1+x)^2}_{\to 0^+}}\Big) = -\infty$$

So we have a VA of $x=-1$, in both directions $f(x) \to -\infty$ as $x \to -1$.

2° Horizontal Asymptotes:

$$\lim_{x\to -\infty} \Big(1 + \frac{x \to -\infty}{(1+x)^2 \to +\infty}\Big) = 1 + \lim_{x\to \infty} \frac{\frac{d}{dx}x}{\frac{d}{dx}(1+x)^2} = 1 + \lim_{x\to -\infty} \frac{1 \to 1}{2(1+x)\cdot 1 \to -\infty}$$

$$= 1 + 0^- = 1^-$$

$$\lim_{x\to +\infty} \Big(1 + \frac{x \to +\infty}{(1+x)^2 \to +\infty}\Big) = 1 + \lim_{x\to \infty} \frac{\frac{d}{dx}x}{\frac{d}{dx}(1+x)^2} = 1 + \lim_{x\to +\infty} \frac{1 \to 1}{2(1+x)\cdot 1 \to +\infty} = 1 + 0^+ = 1^+$$

So we have a HA of $y=1$ in both directions, approached from below as $x \to -\infty$ and from above as $x \to +\infty$.

3° Slope (increase/decrease & max/min)

$$f'(x) = \frac{d}{dx}\left(1 + \frac{x}{(1+x)^2}\right) = 0 + \frac{\left(\frac{d}{dx}x\right)(1+x)^2 - x\,\frac{d}{dx}(1+x)^2}{((1+x)^2)^2}$$

$$= \frac{1\cdot(1+x)^2 - x\cdot 2(1+x)\cdot 1}{(1+x)^4} = \frac{\cancel{(1+x)}\left[(1+x) - 2x\right]}{(1+x)^{\cancel{4}\,3}} = \frac{1-x}{(1+x)^3}$$

is undefined at $x=-1$ (just like $f(x)$) and $=0$ when $x=+1$.

Note that if $x<-1$, $f'(x) = \frac{1-x}{(1+x)^3}$ $\begin{matrix} + & >0 \\ -^3 & <0 \end{matrix}$, <0;

if $-1<x<1$, $f'(x) = \frac{1-x}{(1+x)^3}$ $\begin{matrix} + & >0 \\ + & >0 \end{matrix}$, >0;

if $x>1$, $f'(x) = \frac{1-x}{(1+x)^3}$ $\begin{matrix} - & >0 \\ + & <0 \end{matrix}$, <0.

So $f(x)$ is decreasing on $(-\infty,-1)$ & $(1,\infty)$ & increasing on $(-1,1)$,

so we have a local max at $x=1$

x	$(-\infty,-1)$	-1	$(-1,1)$	1	$(1,\infty)$
$f'(x)$	$-$	undef	$+$	0	$-$
$f(x)$	$\downarrow$	VA $\downarrow$ $-\infty$	$\uparrow$	local max.	$\downarrow$

4° Curvature

$$f''(x) = \frac{d}{dx} f'(x) = \frac{d}{dx}\left(\frac{1-x}{(1+x)^3}\right) = \frac{(-1)(1+x)^3 - (1-x)\,3\,(1+x)^2\cdot 1}{\left[(1+x)^3\right]^2}$$

$$= \frac{(-1)(1+x)^3 - 3(1-x)(1+x)^2}{(1+x)^6} = \frac{\cancel{(1+x)^2}\left[-(1+x) - 3(1-x)\right]}{(1+x)^{\cancel{6}4}}$$

$$= \frac{-1 - x - 3 + 3x}{(1+x)^4} = \frac{2x-4}{(1+x)^4}$$

undefined at $x=-1$

$=0$ when $x=2$

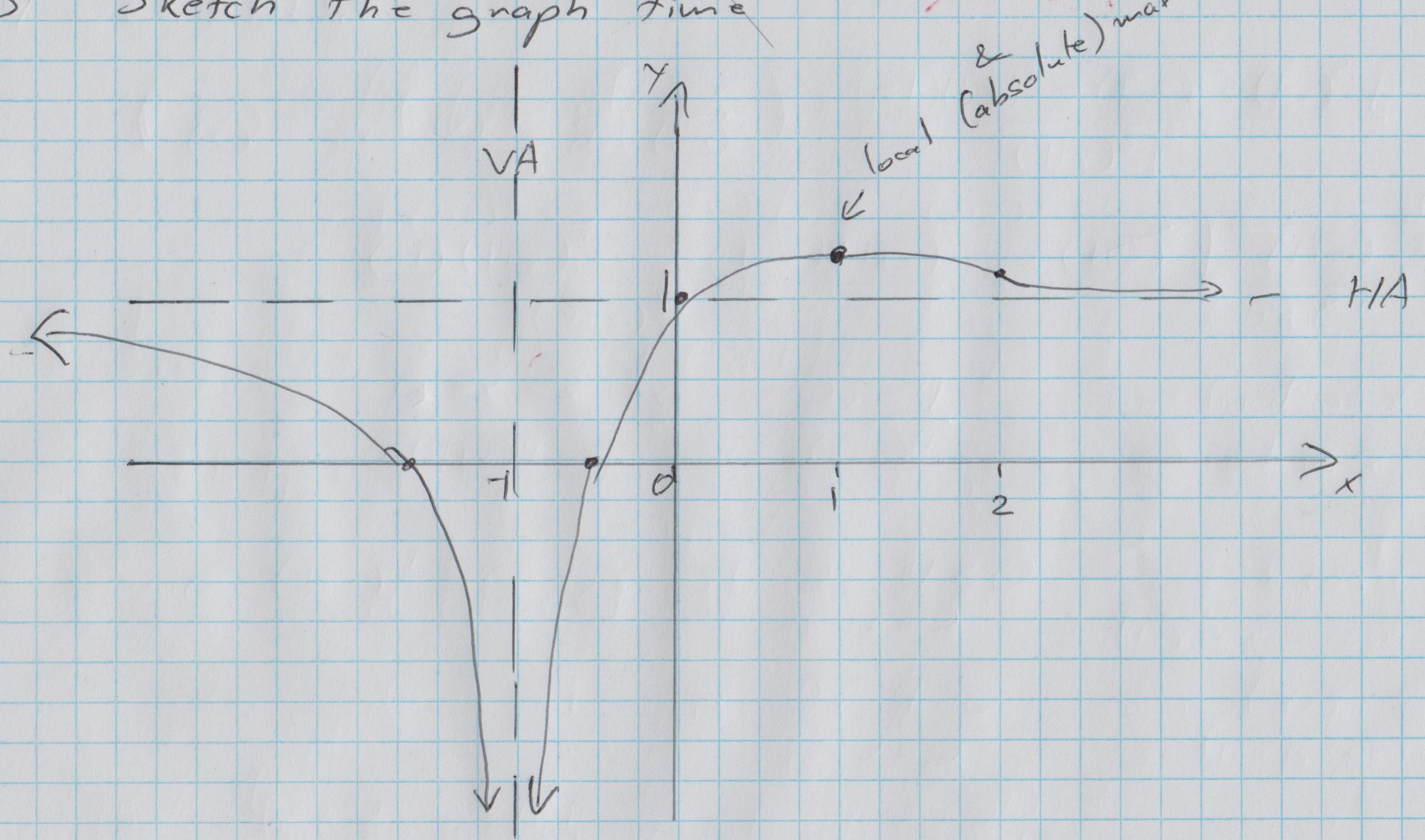
Since $(1+x)^4 \geq 0$, so if $2x-4<0$, so is $f''(x)$,
ie $x<2$ ie $f(x)$ is concave down

if $2x-4>0$, so is $f''(x)$,
ie $x>2$ ie $f(x)$ is concave up.

So we have an inflection point at $x=2$.

x	$(-\infty,-1)$	-1	$(-1,2)$	1	$(1,\infty)$
$f''(x)$	$-$	undef.	~~+~~ $-$	0	$+$
$f(x)$	$\cap$	VA	$\cap$	inflection point	$\cup$

5° Sketch the graph time



$f(2) = 1 + \frac{2}{(1+2)^2} = 1 + \frac{2}{9}$

$f(1) = 1 + \frac{1}{(1+1)^2} = 1 + \frac{1}{4} = \frac{5}{4}$

Have a good Reading Week!