Curve Sketching, ie Qualitative Analysis

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We'll use what we know (and learn a bit more) to analyze the graph of a function.

eg We'll try to find out everything we can about the graph of $f(x) = \frac{x}{1+x^2}$ without actually plotting it. (And then we will sketch the graph using this information.)

0° Intercepts: y-intercept: Plug in $x=0$ – $f(0) = \frac{0}{1+0^2} = 0$, so the y-intercept is at $y=0$.

x-intercepts: Solve $0 = f(x)$ for x, ie $0 = \frac{x}{1+x^2}$

Since $1+x^2 \neq 0$ for all x, $x = 0 \cdot (1+x^2) = 0$; so the only x-intercept is at $x=0$, which happens to be the y-intercept

1° <u>Vertical asymptotes</u>: There are none since $f(x) = \frac{x}{1+x^2}$ is defined and continuous for all x, because $1+x^2 \geq 1 > 0$ for all x.

2° <u>Horizontal asymptotes</u>: $\lim_{x \to -\infty} \frac{x^{\to -\infty}}{1+x^2_{\to +\infty}} = \lim_{x\to -\infty} \frac{\frac{d}{dx}x}{\frac{d}{dx}(1+x^2)} = \lim_{x \to -\infty} \frac{1^{\to 1}}{2x_{\to -\infty}}$

$= 0^-$

$\lim_{x\to+\infty} \frac{x}{1+x^2} = \lim_{x\to+\infty} \frac{x}{1+x^2} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}} = \lim_{x\to+\infty} \frac{\frac{1}{x}^{\to 0}}{\underbrace{\frac{1}{x^2}}_{0} + 1_{\to 1}} = 0^+$

So we have a HA of $y=0$ in both directions, approached from below as $x \to -\infty$ and from above as $x \to +\infty$.

3° <u>Intervals of increase & decrease & local maxima & minima.</u> ("Slope")

$f(x) = \frac{x}{1+x^2}$ $\quad f'(x) = \frac{d}{dx}\left(\frac{x}{1+x^2}\right) = \frac{\left[\frac{d}{dx}x\right](1+x^2) - x\left[\frac{d}{dx}(1+x^2)\right]}{(1+x^2)^2}$

$= \frac{1\cdot(1+x^2) - x(2x)}{(1+x^2)^2} = \frac{1-x^2}{(1+x^2)^2}$

Note that $f'(x)$ (like $f(x)$) is defined and differentiable (3)
(and hence continuous) for all x.

$$f'(x) = \frac{1-x^2}{(1+x^2)^2} = 0 \iff 1-x^2 = 0 \iff x = \pm 1$$

[Since $(1+x^2)^2 > 0$] $> 0 \iff 1-x^2 > 0 \iff x^2 < 1 \iff |x| < 1$

$< 0 \iff 1-x^2 < 0 \iff x^2 > 1 \iff |x| > 1$

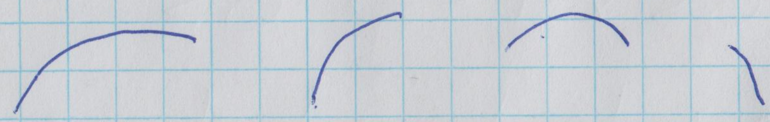
So we have a critical point at $x = \pm 1$, and $f(x)$ is decreasing before $x=-1$, increasing from $x=-1$ to $x=+1$, and decreasing after $x=+1$; and so $x=-1$ is a local minimum and $x=+1$ is a local maximum.

x	$(-\infty, -1)$	-1	$(-1, 1)$	1	$(1, \infty)$
$f'(x)$	$-$	0	$+$	0	$-$
$f(x)$	$\downarrow$	local min	$\uparrow$	local max	$\downarrow$

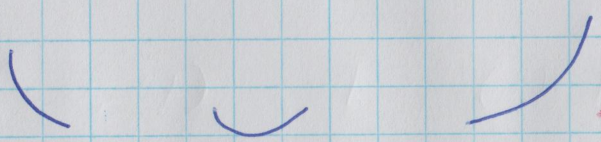
4° Curvature

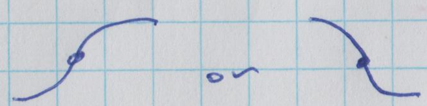
$f''(x)$ tells us if $f'(x)$ is increasing or decreasing or neither

If $f''(x) < 0$, then the derivative is decreasing, so $f(x)$ curves downward (as you go from left to right).

eg  etc "concave down"

If $f''(x) > 0$, then the derivative is increasing, so $f(x)$ curves upward (going from left to right)

eg , etc. "concave up"

The points where $f''(x) = 0$ and we make a transition from concave down to concave up, or vice versa, are called inflection points. eg 

$$f''(x) = \frac{d}{dx} f'(x) = \frac{d}{dx}\left(\frac{1-x^2}{(1+x^2)^2}\right) = \frac{\left[\frac{d}{dx}(1-x^2)\right](1+x^2)^2 - (1-x^2)\left[\frac{d}{dx}(1+x^2)^2\right]}{\left[(1+x^2)^2\right]^2}$$

$$= \frac{[-2x]\,(1+x^2)^2 - (1-x^2)\left[2(1+x^2)\cdot\frac{d}{dx}(1+x^2)\right]}{(1+x^2)^4}$$

$$= \frac{-2x\,(1+x^2)^{\cancel{2}\,1} - (1-x^2)\,2\cdot\cancel{(1+x^2)}\cdot 2x}{(1+x^2)^{\cancel{4}\,3}}$$

$$= \frac{-2x - 2x^3 - 4x + 4x^3}{(1+x^2)^3} = \frac{-6x + 2x^3}{(1+x^2)^3} = \frac{2x(x^2-3)}{(1+x^2)^3}$$

Since $(1+x^2)^3 \geq 1 > 0$, $f''(x)$ [like $f(x)$ & $f'(x)$] is defined for all x.

$f''(x) = 0 \iff 2x(x^2-3) = 0 \iff x = 0$ or $x^2 - 3 = 0$

$\iff x = 0$ or $x = \sqrt{3}$ or $x = -\sqrt{3}$

$f''(x) > 0 \iff 2x(x^2-3) > 0 \iff (x > 0$ and $x^2 > 3)$

or $(x < 0$ and $x^2 < 3)$

$\iff (x > 0$ and $|x| > \sqrt{3})$

or $(x < 0$ and $|x| < 3)$

$\iff x > \sqrt{3}$ or $-\sqrt{3} < x < 0$

Similarly

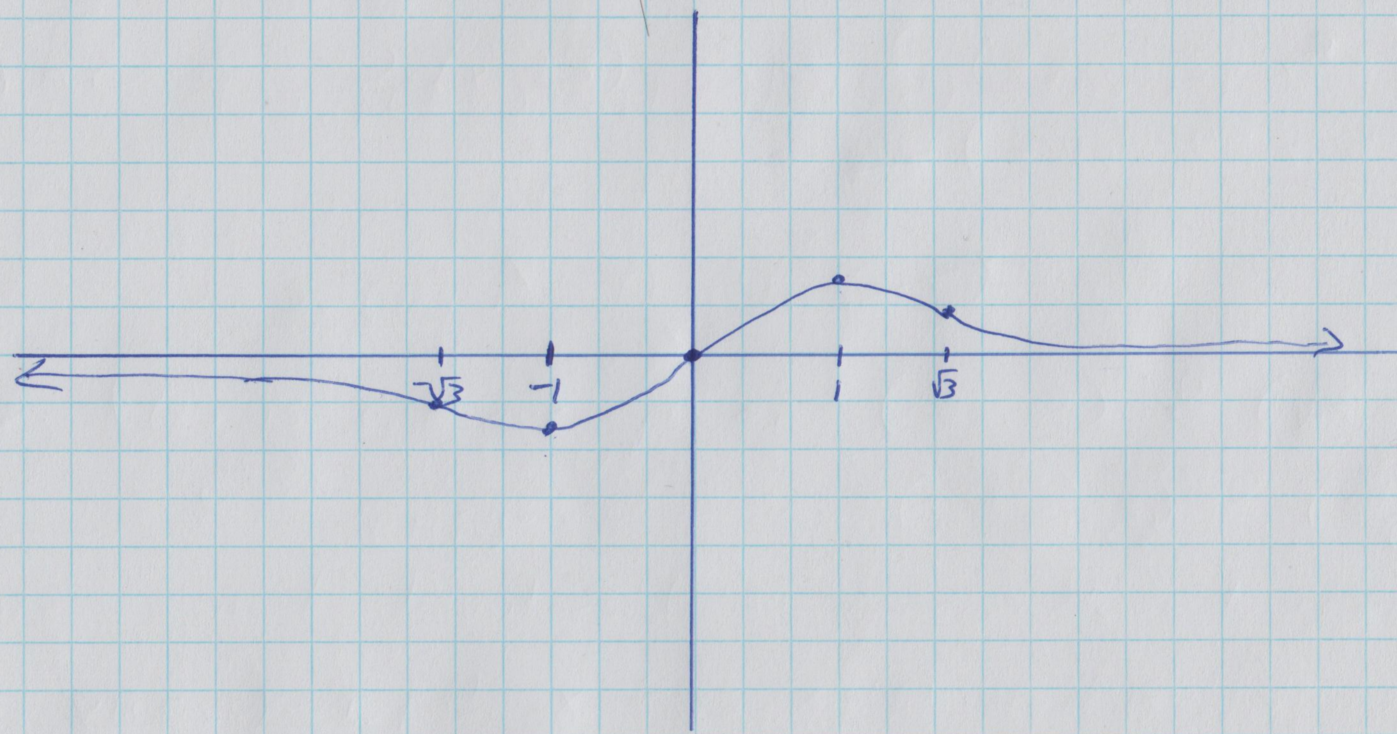
$$f''(x) < 0 \iff 2x(x^2-3) < 0 \iff (x<0 \ \& \ x^2 > 3)$$
$$\text{or } (x>0 \ \& \ x^2 < 3)$$
$$\iff x < -\sqrt{3} \quad \text{or} \quad 0 < x < \sqrt{3}$$

Thus $f(x)$ is concave up on $(-\sqrt{3}, 0)$ & $(\sqrt{3}, \infty)$
& concave down on $(-\infty, -\sqrt{3})$ & $(0, \sqrt{3})$.

Since we make transitions from concave down to concave up or vice versa at $x = -\sqrt{3}$, $x = 0$, and $x = \sqrt{3}$, we have inflection points at all three.

x	$(-\infty, -\sqrt{3})$	$-\sqrt{3}$	$(-\sqrt{3}, 0)$	0	$(0, \sqrt{3})$	$\sqrt{3}$	$(\sqrt{3}, \infty)$
$f''(x)$	$-$	0	$+$	0	$-$	0	$+$
$f(x)$	$\frown$	infl. pt.	$\smile$	infl. pt.	$\frown$	infl. pt.	$\smile$

5° Sketch of the graph of $f(x) = \frac{x}{1-x^2}$



$$f(-1) = \frac{-1}{1+(-1)^2} = -\frac{1}{2} \qquad f(-\sqrt{3}) = \frac{-\sqrt{3}}{1+(-\sqrt{3})^2} = \frac{-\sqrt{3}}{1+3} = -\frac{\sqrt{3}}{4}$$

$$f(1) = \frac{1}{1+1^2} = \frac{1}{2} \qquad f(\sqrt{3}) = \frac{\sqrt{3}}{1+(\sqrt{3})^2} = \frac{\sqrt{3}}{1+3} = \frac{\sqrt{3}}{4}$$

Note that it is obvious from the sketch that the local min & local max are also absolute.