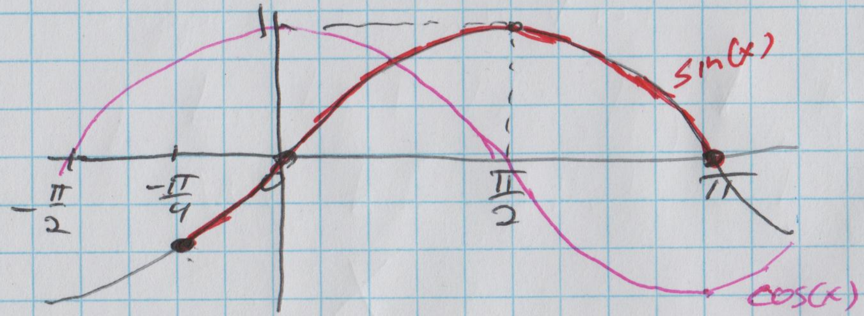
2020-10-13

Maxima and Minima II - Examples

1° $f(x) = \sin(x)$ on $\left[-\frac{\pi}{4}, \pi\right]$

including endpoints

Find the local and absolute maxima and minima.



From the graph the only local min is at $x = \frac{\pi}{2}$ ($\sin(\frac{\pi}{2}) = 1$) and this is also the absolute maximum of $\sin(x)$ on $\left[-\frac{\pi}{4}, \pi\right]$.
The minimum occurs at $x = -\frac{\pi}{4}$ (one of the endpoints) and $\sin(-\frac{\pi}{4}) = -\frac{1}{\sqrt{2}}$.

1° Find the critical point(s), if any, in the interval.

$$\frac{d}{dx}\sin(x) = \cos(x)$$

$$= 0 \quad \text{when } x = \frac{\pi}{2} \pm n\pi$$

The only such point in $\left[-\frac{\pi}{4}, \pi\right]$ is $x = \frac{\pi}{2}$.

2° Since the interval includes both endpoints, we compare the values of the function at the critical points and the endpoints

$\sin(\frac{\pi}{2}) = 1$ max

$\sin(-\frac{\pi}{4}) = -\frac{1}{\sqrt{2}}$ min

$\sin(\pi) = 0$ neither

2° $f(x) = e^{-x^2}$ (a scaled bell curve)

(actual standard normal curve is $g(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$)

Find the maxima and minima on $(-\infty, \infty)$ of $f(x)$.

1° Find the critical points.

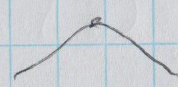
$$\frac{d}{dx} e^{-x^2} = e^{-x^2} \cdot \frac{d}{dx}(-x^2) = e^{-x^2} \cdot (-2x) = \overbrace{-2}^{\neq 0} x \overbrace{e^{-x^2}}^{>0}$$

$= 0$ exactly when $x = 0$

2° Check to see if these are local max or local min (or neither)

$f'(x) = \overbrace{-2x}^{<0} \overbrace{e^{-x^2}}^{>0}$ $\quad > 0$ if $x < 0$

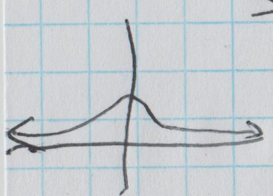
$\quad < 0$ if $x > 0$



so $f(x)$ is increasing before $x = 0$ & decreasing after

so $x = 0$ is a local max, $f(0) = e^{-0^2} = e^0 = 1$.

3° We don't have endpoints so we can limits to see what happens "as we approach the endpoints".



$$\lim_{x \to \pm\infty} e^{-x^2} = 0^+$$

So there is no minimum, but 0 is the greatest lower bound.

as $x \to \pm\infty$, $x^2 \to +\infty$, $-x^2 \to -\infty$, so $e^{-x^2} \to 0^+$

3° $g(x)=\frac{x^2}{1+x^2}$ on $[-15, 16)$ Find the local & abs. maxima & minima of $f(x)$ on the interval. ③

$\uparrow$ included $\uparrow$ excluded

$1+x^2 \geq 1 > 0$ so it's defined for all x

$= \{x \mid -15 \leq x < 16\}$

1° Find the critical point(s).

$$g'(x) = \frac{d}{dx}\left(\frac{x^2}{1+x^2}\right) = \frac{\left(\frac{d}{dx}x^2\right)(1+x^2) - x^2\left(\frac{d}{dx}(1+x^2)\right)}{(1+x^2)^2}$$

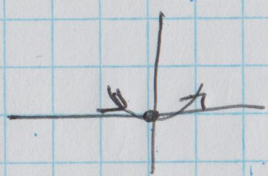
$$= \frac{2x(1+x^2) - x^2(2x)}{(1+x^2)^2} = \frac{2x + \cancel{2x^3} - \cancel{2x^3}}{(1+x^2)^2} = \frac{2x}{(1+x^2)^2}$$

$= 0$ exactly when $x = 0$

[Note that it's defined for all x, since $(1+x^2)^2 \geq 1 > 0$]

2° Check to see if any such points are local maxima or minima.

$$g'(x) = \frac{2x \gtrless 0}{(1+x^2)^2 > 0}$$ depending on whether $x \gtrless 0$.



$g'(x) < 0$ when $x < 0$ & > 0 when $x > 0$.

so $g(0) = 0$ is a local minimum.

3° Check the endpoints: Included endpoint: evaluate $g(x)$: $g(-15) = \frac{225}{226}$

Excluded endpoint: take the limit $\lim_{x \to 16} \frac{x^2}{1+x^2} = \frac{256}{257} > \frac{225}{226} = g(-15)$

On the given interval,

$g(0) = 0$ is the smallest, so it's the absolute minimum &

$g(16) = \frac{256}{257}$ is the least upper bound on $[-15, 16)$, but there is no absolute maximum.

4° $h(x) = \frac{x^2}{1-x^2}$ on $(-\infty, \infty)$ Find all local & abs. maxima & minima. ④

1° Find the critical points.

$$h'(x) = \frac{d}{dx}\left(\frac{x^2}{1-x^2}\right) = \frac{\left(\frac{d}{dx}(x^2)\right)(1-x^2) - (x^2)\frac{d}{dx}(1-x^2)}{(1-x^2)^2}$$

$$= \frac{2x(1-x^2) - x^2(-2x)}{(1-x^2)^2} = \frac{2x - \cancel{2x^3} + \cancel{2x^3}}{(1-x^2)^2} = \frac{2x}{(1-x^2)^2}$$

Note that $1-x^2 = 0$ when $x = \pm 1$, so both $h(x)$ and $h'(x)$ are undefined at $x = \pm 1$.

$h'(x) = \frac{2x}{(1-x^2)^2} = 0$ exactly when $x = 0$

We also need to consider what happens at points where $g(x)$ &/or $g'(x)$ are undefined, so such points are often included among the critical points.

2° Check what happens at the critical points

$h'(x) = \frac{2x}{(1-x^2)^2}$ ± depending on whether x is ± .
>0

$h'(x) < 0$ when $x < 0$ so $h(x)$ is decreasing

$\therefore$ $x = 0$ is a local min. $h'(x) > 0$ when $x > 0$ so $h(x)$ is increasing

Also check what happens at points where $h(x)$ is undefined: (5)

Take limits from each direction:

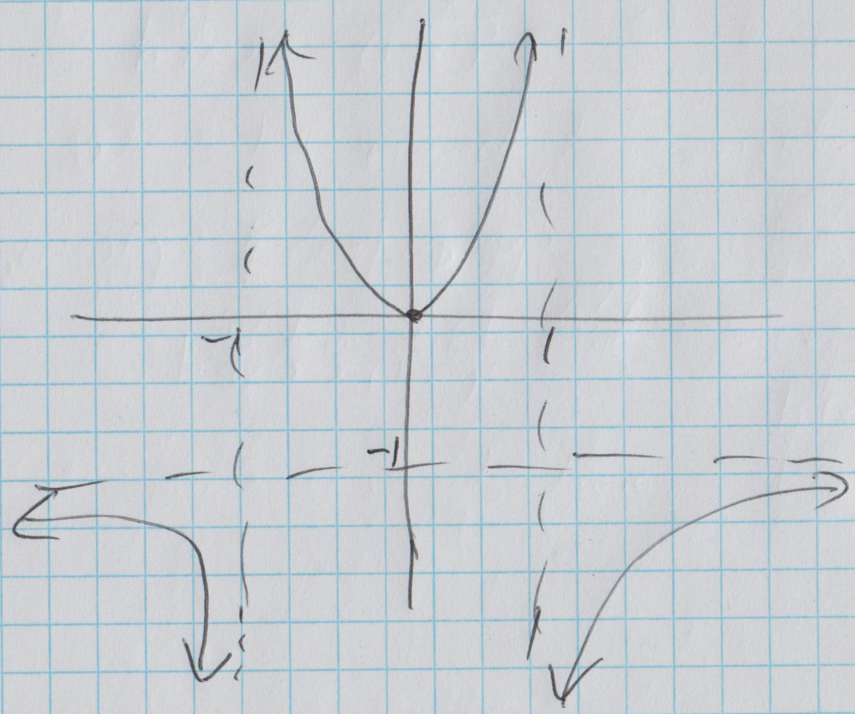
$\lim_{x \to -1^-} \frac{x^2}{1-x^2} = \frac{1}{0^-} = -\infty$ $\quad$ $\lim_{x \to -1^+} \frac{x^2}{1-x^2} = \frac{1}{0^+} = +\infty$

$x < -1$, $|x| > 1$, so $1 - x^2 < 0$ $\quad$ $x > -1$, $|x| < 1$, so $1 - x^2 > 0$

$\lim_{x \to +1^-} \frac{x^2}{1-x^2} = \frac{1}{0^+} = +\infty$ $\quad$ $\lim_{x \to +1^+} \frac{x^2}{1-x^2} = \frac{1}{0^-} = -\infty$

$x < 1$, so $1 - x^2 > 0$ $\quad$ $x > 1$ so $1 - x^2 < 0$

Because you have VA in both direction, $h(x)$ has no absolute max or min on $(-\infty, \infty)$



∴ No abs max or min due to VA that go in both direction, but have a local min. at $x = 0$.