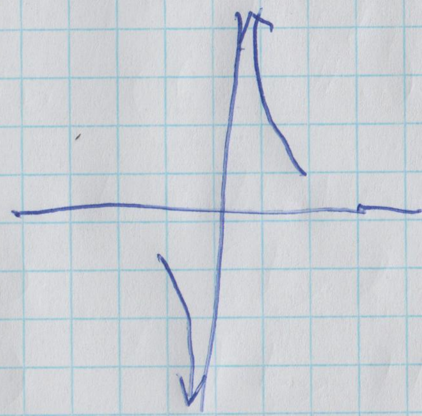
2020-10-05 ①

Asymptotes

[Held off on this until we had l'Hôpital's Rule.]

Two kinds: Vertical asymptotes

<u>es</u> $f(x) = \frac{1}{x}$

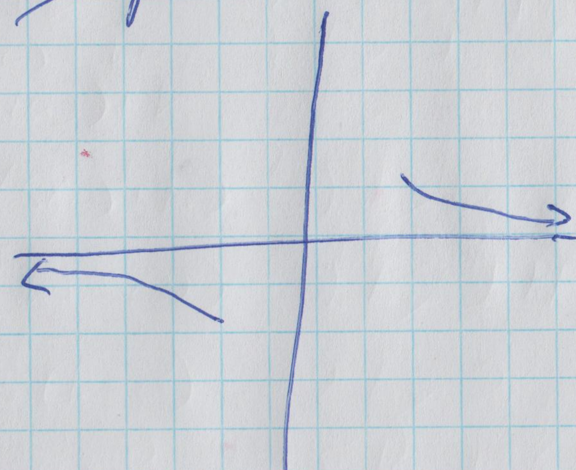


$\lim_{x \to 0^+} \frac{1}{x} = +\infty$

$\lim_{x \to 0^-} \frac{1}{x} = -\infty$

Horizontal asymptotes

<u>es</u> $f(x) = \frac{1}{x}$



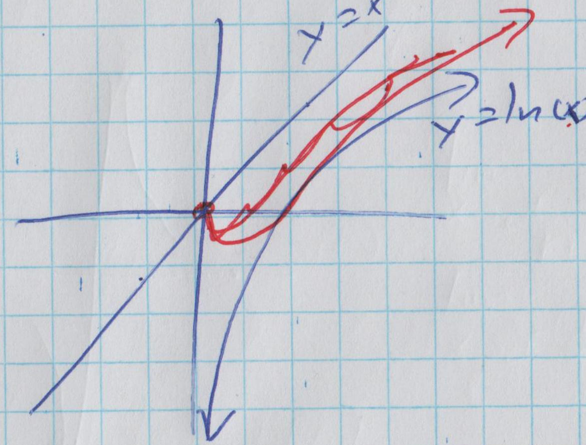
$\lim_{x \to +\infty} \frac{1}{x} = 0^+$

$\lim_{x \to -\infty} \frac{1}{x} = 0^-$

For vertical asymptotes we usually have them at points where the (expression defining the) function is undefined.

<u>es</u> $f(x) = x\ln(x)$

Does this have a vertical asymptote as $x \to 0^+$?



$$\lim_{x\to 0^+} \overset{\nearrow 0^+}{x}\,\overset{\nearrow -\infty}{\ln(x)}$$

$$= \lim_{x\to 0^+} \frac{\ln(x) \to -\infty}{\frac{1}{x} \to +\infty} \overset{\text{l'Hôpital's Rule}}{=} \lim_{x\to 0^+} \frac{\frac{d}{dx}\ln(x)}{\frac{d}{dx}\left(\frac{1}{x}\right)} = \lim_{x\to 0^+} \frac{\frac{1}{x}}{\frac{-1}{x^2}} = \lim_{x\to 0^+} \frac{1}{x}\cdot\frac{x^2}{-1}$$

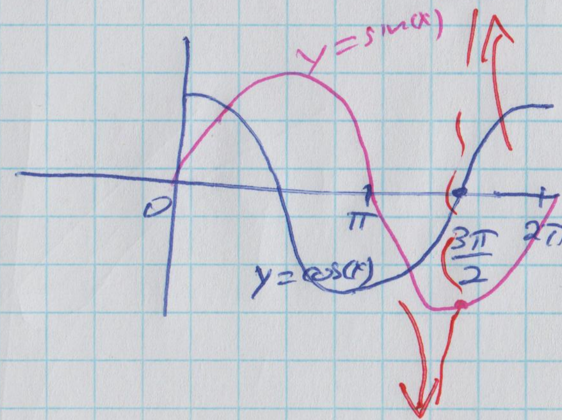
$$= \lim_{x\to 0^+}\left(-\frac{x^2}{x}\right) = \lim_{x\to 0^+}(-x) = -0^+ = 0^-$$

$\therefore$ This does <u>not</u> have a vertical asymptote at $x=0$.

eg $f(x) = \sec(x) = \dfrac{1}{\cos(x)}$ at $x = \dfrac{3\pi}{2}$

[Note that $\cos(x) = 0$ when $x = \frac{\pi}{2} \pm n\pi$.]

To check for a vertical asymptote at $x = \frac{3\pi}{2}$ take the one-sided limit from each side the function is defined on.
[In this case, it's both sides.]



$$\lim_{x\to\frac{3\pi}{2}^-} \sec(x) = \lim_{x\to\frac{3\pi}{2}^-} \frac{1}{\cos(x)} \begin{matrix}\to 1\\ \to 0^-\end{matrix} = -\infty$$

$$\lim_{x\to\frac{3\pi}{2}^+} \sec(x) = \lim_{x\to\frac{3\pi}{2}^+} \frac{1}{\cos(x)} \begin{matrix}\to 1\\ \to 0^+\end{matrix} = +\infty$$

eg $f(x) = \left(x - \frac{3\pi}{2}\right)\sec(x)$ at $x = \frac{3\pi}{2}$

l'Hôpital's Rule

$$\lim_{x\to\frac{3\pi}{2}^-} \left(x - \frac{3\pi}{2}\right)\sec(x) = \lim_{x\to\frac{3\pi}{2}^-} \frac{x - \frac{3\pi}{2}}{\cos(x)} \begin{matrix}\to 0^-\\ \to 0^-\end{matrix} = \lim_{x\to\frac{3\pi}{2}^-} \frac{\frac{d}{dx}\left(x - \frac{3\pi}{2}\right)}{\frac{d}{dx}\cos(x)}$$

$$= \lim_{x\to\frac{3\pi}{2}^-} \frac{1}{-\sin(x)} \begin{matrix}\to 1\\ \to -(-1) = 1\end{matrix} = \frac{1}{1} = 1$$

Similarly, $\lim_{x\to\frac{3\pi}{2}^+} \left(x - \frac{3\pi}{2}\right)\sec(x) = 1.$

Thus $f(x) = \left(x - \frac{3\pi}{2}\right)\sec(x)$ does <u>not</u> have a vertical asymptote at $x = \frac{3\pi}{2}$

eg $\lim_{x\to 0^+} \frac{e^x-1 \to 0^+}{x \to 0^+} = \lim_{x\to 0^+} \frac{\frac{d}{dx}(e^x-1)}{\frac{d}{dx}(x)} = \lim_{x\to 0^+} \frac{e^x-0}{1} = e^0 = 1$

$\lim_{x\to 0^-} \frac{e^x-1 \to 0^-}{x \to 0^-} = \lim_{x\to 0^-} \text{ — '' — } = \lim_{x\to 0^-} \text{ — '' — } = e^0 = 1$

so this has no vertical asymptotes at $x=0$

[Notice that $\frac{e^x-1}{x}$ is undefined at $x=0$]

eg $\lim_{x\to 0^+} \frac{x \to 0^+}{e^{x^2}-1 \to 0^+} = \lim_{x\to 0^+} \frac{\frac{d}{dx}(x)}{\frac{d}{dx}(e^{x^2}-1)} = \lim_{x\to 0^+} \frac{1}{e^{x^2}\cdot\frac{d}{dx}(x^2) - 0}$

$= \lim_{x\to 0^+} \frac{1}{\underset{\downarrow 0^+}{2x}\,\underset{\downarrow 1}{e^{x^2}}} = +\infty$

$\lim_{x\to 0^-} \frac{x \to 0^-}{e^{x^2}-1 \to 0^-} = \lim_{x\to 0^-} \frac{\frac{d}{dx}(x)}{\frac{d}{dx}(e^{x^2}-1)} = \lim_{x\to 0^-} \frac{1}{e^{x^2}\cdot\frac{d}{dx}(x^2) - 0}$

$= \lim_{x\to 0^-} \frac{1 \to 1}{2xe^{x^2} \to 0^-} = -\infty$

So we have a vertical asymptote (on each side of) $x=0$

(& going in opposite directions on each side).

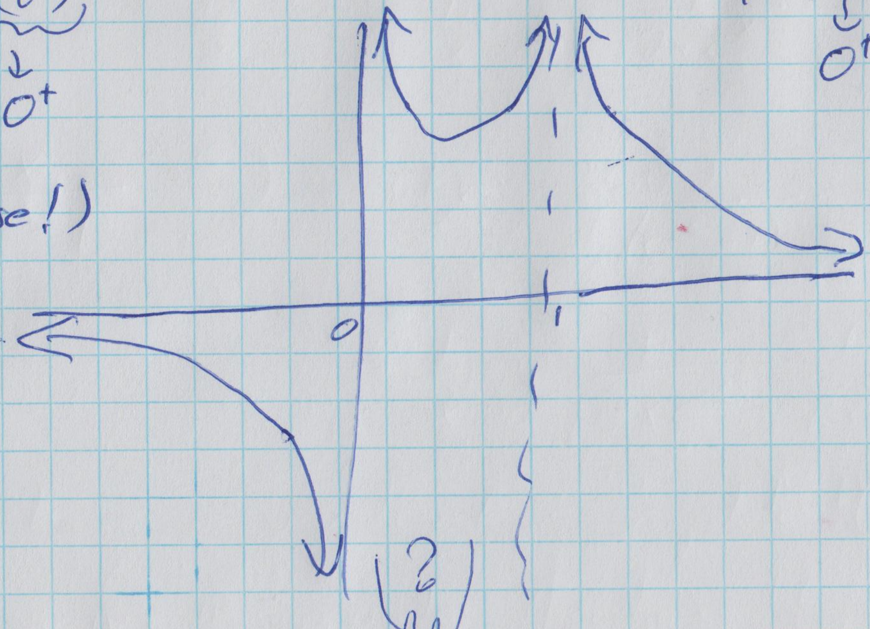
eg $f(x) = \dfrac{x^2+1}{x^3-2x^2+x} = \dfrac{x^2+1}{x(x^2-2x+1)} = \dfrac{x^2+1}{x(x-1)^2}$

is undefined at $x=0$ and $x=1$.

Vertical asymptotes at these points?

$$\lim_{x\to 0^-} \frac{\overbrace{(x^2+1)}^{\to 1}}{\underbrace{x}_{0^-}\underbrace{(x-1)^2}_{1}} = -\infty \qquad \lim_{x\to 0^+} \frac{\overbrace{(x^2+1)}^{\to 1}}{\underbrace{x}_{0^+}\underbrace{(x-1)^2}_{1}} = +\infty$$

$$\lim_{x\to 1^-} \frac{\overbrace{(x^2+1)}^{\to 2}}{\underbrace{x}_{1}\underbrace{(x-1)^2}_{\underbrace{(0^-)^2}_{0^+}}} = +\infty \qquad \lim_{x\to 1^+} \frac{\overbrace{(x^2+1)}^{\to 1}}{\underbrace{x}_{1}\underbrace{(x-1)^2}_{\underbrace{(0^+)^2}_{0^+}}} = +\infty$$



Similarly (exercise!)

$$\lim_{x\to +\infty} \frac{x^2+1}{x(x-1)^2} = 0^+$$

Horizontal Asymp?

$$\lim_{x\to -\infty} \frac{\overbrace{x^2+1}^{\to\infty}}{\underbrace{x}_{-\infty}\underbrace{(x-1)^2}_{\infty}} = \lim_{x\to -\infty} \frac{x^2+1}{x^3-2x^2+x}$$

$$= \lim_{x\to -\infty} \frac{\frac{d}{dx}(x^2+1)}{\frac{d}{dx}(x^3-2x^2+x)}$$

$$= \lim_{x\to -\infty} \frac{2x}{3x^2-4x+1} \quad \begin{matrix}\to -\infty \\ \to +\infty\end{matrix}$$

$$= \lim_{x\to -\infty} \frac{\frac{d}{dx}(2x)}{\frac{d}{dx}(3x^2-4x+1)}$$

$$= \lim_{x\to -\infty} \frac{\overbrace{2}^{\to 2}}{\underbrace{6x-4}_{\to -\infty}} = 0^-$$