Derivatives VI - L'Hôpital's Rule

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or, using derivatives to help with indeterminate limits.

$$\lim_{x \to a} \frac{f(x)}{g(x)} \quad \begin{matrix} \to 0 \\ \to 0 \end{matrix} \quad \text{or} \quad \begin{matrix} \to \pm\infty \\ \to \pm\infty \end{matrix} \qquad \frac{?}{0}$$

L'Hôpital's Rule:

(Originally published in 1696 in a book that was anonymous, but written by L'Hôpital, and based in part on work by Johann Bernoulli.)

If $f(x)$ and $g(x)$ are differentiable functions and $\lim_{x \to a} \frac{f'(x)}{g'(x)}$ exists and $= L$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = L.$$

eg 0° $$\lim_{x \to 0} \frac{x^2 \to 0}{\sin(x) \to 0} \overset{\text{by L'Hôpital's Rule}}{=} \lim_{x \to 0} \frac{\frac{d}{dx} x^2}{\frac{d}{dx} \sin(x)} = \lim_{x \to 0} \frac{2x \to 0}{\cos(x) \to 1}$$

$$= \frac{2 \cdot 0}{\cos(0)} = \frac{2 \cdot 0}{1} = 0$$

(2)

1° $\lim\limits_{x\to\infty} \dfrac{x}{e^x}$ $\begin{matrix}\to\infty\\ \to\infty\end{matrix}$ $\overset{\text{l'Hôpital's Rule}}{=} \lim\limits_{x\to\infty} \dfrac{\frac{d}{dx}x}{\frac{d}{dx}e^x} = \lim\limits_{x\to\infty} \dfrac{1}{e^x}$ $\begin{matrix}\to 1\\ \to\infty\end{matrix}$

$= 0$

2° $\lim\limits_{x\to\infty} \dfrac{x^3}{e^x}$ $\begin{matrix}\to\infty\\ \to\infty\end{matrix}$ $= \lim\limits_{x\to\infty} \dfrac{\frac{d}{dx}x^3}{\frac{d}{dx}e^x} = \lim\limits_{x\to\infty} \dfrac{3x^2}{e^x}$ $\begin{matrix}\to\infty\\ \to\infty\end{matrix}$

$= \lim\limits_{x\to\infty} \dfrac{\frac{d}{dx}(3x^2)}{\frac{d}{dx}e^x} = \lim\limits_{x\to\infty} \dfrac{6x}{e^x}$ $\begin{matrix}\to\infty\\ \to\infty\end{matrix}$

$= \lim\limits_{x\to\infty} \dfrac{\frac{d}{dx}(6x)}{\frac{d}{dx}e^x} = \lim\limits_{x\to\infty} \dfrac{6}{e^x}$ $\begin{matrix}\to 6\\ \to\infty\end{matrix}$ $= 0$

3° $\lim\limits_{x\to 1} \dfrac{x-1}{\ln(x)}$ $\begin{matrix}\to 0\\ \to 0\end{matrix}$ $= \lim\limits_{x\to 1} \dfrac{\frac{d}{dx}(x-1)}{\frac{d}{dx}\ln(x)} = \lim\limits_{x\to 1} \dfrac{1}{\frac{1}{x}} = \lim\limits_{x\to 1} 1\cdot\dfrac{x}{1} = \lim\limits_{x\to 1} x = 1$

4° $\lim\limits_{x\to 1} \dfrac{x-1}{\sqrt{x}-1}$ $\begin{matrix}\to 0\\ \to 0\end{matrix}$ $= \lim\limits_{x\to 1} \dfrac{\frac{d}{dx}(x-1)}{\frac{d}{dx}(\sqrt{x}-1)} = \lim\limits_{x\to 1} \dfrac{1}{\frac{1}{2\sqrt{x}}} = \lim\limits_{x\to 1} 2\sqrt{x} = 2\cdot\sqrt{1} = 2$

without l'Hôpital: $= \lim\limits_{x\to 1} \dfrac{\cancel{(\sqrt{x}-1)}(\sqrt{x}+1)}{\cancel{\sqrt{x}-1}} = \lim\limits_{x\to 1} (\sqrt{x}+1) = \sqrt{1}+1 = 1+1 = 2$

5° $\lim\limits_{x\to 2} \dfrac{(\sqrt{x}-\sqrt{2})^{12}}{x^6-64} \begin{matrix}\to 0\\ \to 0\end{matrix} = \lim\limits_{x\to 2} \dfrac{\frac{d}{dx}(\sqrt{x}-\sqrt{2})^{12}}{\frac{d}{dx}(x^6-64)}$

$= \lim\limits_{x\to 2} \dfrac{12(\sqrt{x}-\sqrt{2})^{11}\cdot\frac{d}{dx}(\sqrt{x}-\sqrt{2})}{6x^5} = \lim\limits_{x\to 2} \dfrac{12(\sqrt{x}-\sqrt{2})^{11}\cdot\frac{1}{2\sqrt{x}}}{6x^5}$

$= \dfrac{12(\sqrt{2}-\sqrt{2})^{11}\cdot\frac{1}{2\sqrt{2}}}{6\cdot 2^5} = \dfrac{12\cdot 0^{11}\cdot\frac{1}{2\sqrt{2}}}{6\cdot 2^5} = \dfrac{0}{6\cdot 2^5} = 0$

6° $\lim\limits_{x\to\infty}\left(\underset{\downarrow\atop\infty}{\sqrt{x+1}} - \underset{\downarrow\atop\infty}{\sqrt{x-1}}\right) = \lim\limits_{x\to\infty}\left(\sqrt{x+1}-\sqrt{x-1}\right)\cdot\dfrac{\sqrt{x+1}+\sqrt{x-1}}{\sqrt{x+1}+\sqrt{x-1}}$

$= \lim\limits_{x\to\infty} \dfrac{(\sqrt{x+1})^2-(\sqrt{x-1})^2}{\sqrt{x+1}+\sqrt{x-1}} = \lim\limits_{x\to\infty} \dfrac{(x+1)-(x-1)}{\sqrt{x+1}+\sqrt{x-1}}$

$= \lim\limits_{x\to\infty} \dfrac{2}{\sqrt{x+1}+\sqrt{x-1}} \begin{matrix}\to 2\\ \to\infty\end{matrix} = 0$

Notice that trying to put this into a form where l'Hopital's Rule might apply puts in a form where you don't need the Rule...