Derivatives IV - Examples (and building a basic library)

2020-09-28 ①

Our library of common basic derivatives includes:

1° $\frac{d}{dx} c = 0$ (c a constant)

2° $\frac{d}{dx} x^n = n x^{n-1}$ (n could be any real number)

3° $\frac{d}{dx} e^x = e^x$

4° $\frac{d}{dx} \ln(x) = \frac{1}{x}$

5° $\frac{d}{dx} \arctan(x) = \frac{1}{1+x^2}$

6° $\frac{d}{dx} \sin(x) = \cos(x)$

7° $\frac{d}{dx} \cos(x) = -\sin(x)$

8° $\frac{d}{dx} \tan(x) = \sec^2(x) = \frac{1}{\cos^2(x)}$

9° $\frac{d}{dx} \sec(x) = \frac{d}{dx}\left(\frac{1}{\cos(x)}\right) = \frac{d}{dx} (\cos(x))^{-1}$

$= -1 \cdot (\cos(x))^{-2} \cdot \frac{d}{dx} \cos(x)$

$= \frac{\cancel{-}1}{\cos^2(x)} \cdot (\cancel{-}\sin(x))$

$= \frac{\sin(x)}{\cos^2(x)} = \frac{\sin(x)}{\cos(x)} \cdot \frac{1}{\cos(x)} = \tan(x)\sec(x)$

$$0 = \frac{d}{dx}(1) = \frac{d}{dx}\left(\sin^2(x) + \cos^2(x)\right)$$

$$= \frac{d}{dx}\sin^2(x) + \frac{d}{dx}\cos^2(x)$$

$$= 2\sin(x)\cdot\frac{d}{dx}\sin(x) + 2\cos(x)\cdot\frac{d}{dx}\cos(x)$$

$$= 2\sin(x)\cos(x) + 2\cos(x)\cdot\frac{d}{dx}\cos(x)$$

$$= 2\cos(x)\left[\sin(x) + \frac{d}{dx}\cos(x)\right]$$

$$\Rightarrow 0 = \sin(x) + \frac{d}{dx}\cos(x)$$

$$\Rightarrow \frac{d}{dx}\cos(x) = -\sin(x)$$

10° $\frac{d}{dx} a^x = \frac{d}{dx} \left(e^{\ln(a)}\right)^x = \frac{d}{dx} e^{\ln(a) \cdot x}$

$(a > 0)$

$a = e^{\ln(a)}$

$$= e^{\ln(a) \cdot x} \cdot \frac{d}{dx}\left(\ln(a) \cdot x\right)$$

$$= a^x \cdot \ln(a) \cdot \frac{d}{dx} x$$

$$= a^x \cdot \ln(a) \cdot 1$$

$$= \ln(a) \cdot a^x$$

11° $\frac{d}{dx} \log_a(x) = \frac{d}{dx}\left(\frac{\ln(x)}{\ln(a)}\right) = \frac{1}{\ln(a)} \frac{d}{dx} \ln(x)$

$(a > 0)$

$$\boxed{\log_a(x) = \frac{\ln(x)}{\ln(a)}}$$

$$= \frac{1}{\ln(a)} \cdot \frac{1}{x} = \frac{1}{\ln(a) \cdot x}$$

12° $\frac{d}{dx} \underbrace{\arcsin(x)}_{\sin^{-1}(x)} = \frac{1}{\sqrt{1-x^2}}$

$\sin(\arcsin(x)) = x \Rightarrow \frac{d}{dx} \sin(\arcsin(x)) = \frac{dx}{dx} = 1$

$$= \cos(\arcsin(x)) \cdot \frac{d}{dx} \arcsin(x)$$

$$= \sqrt{1 - \sin^2(\arcsin(x))} \cdot \frac{d}{dx} \arcsin(x) = \sqrt{1-x^2} \cdot \frac{d}{dx} \arcsin(x)$$

"Hyperbolic functions"

11° $\frac{d}{dx}\cosh(x) = \frac{d}{dx}\left(\frac{e^x+e^{-x}}{2}\right) = \frac{1}{2}\left(e^x + e^{-x}\cdot\frac{d}{dx}(-x)\right)$

$\cosh(x) = \frac{e^x+e^{-x}}{2}$

"kosh"

$= \frac{1}{2}\left(e^x + e^{-x}(-1)\right)$

$= \frac{1}{2}\left(e^x - e^{-x}\right) = \frac{e^x - e^{-x}}{2}$

$= \sinh(x)$

"cinch"

12° $\frac{d}{dx}\sinh(x) = \frac{d}{dx}\left(\frac{e^x-e^{-x}}{2}\right) = \frac{e^x - e^{-x}(-1)}{2} = \frac{e^x+e^{-x}}{2} = \cosh(x)$

$\cosh^2(x) - \sinh^2(x) = \left(\frac{e^x+e^{-x}}{2}\right)^2 - \left(\frac{e^x-e^{-x}}{2}\right)^2$

$= \frac{\cancel{(e^x)^2} + 2e^xe^{-x} + \cancel{(e^{-x})^2} - \cancel{(e^x)^2} + 2e^xe^{-x} - \cancel{(e^{-x})^2}}{4}$

$= \frac{\cancel{4}e^xe^{-x}}{\cancel{4}} = e^{x-x} = e^0 = 1$

13° $\frac{d}{dx}\tanh(x) = \frac{d}{dx}\left(\frac{\sinh(x)}{\cosh(x)}\right) = \frac{\left[\frac{d}{dx}\sinh(x)\right]\cosh(x) - \sinh(x)\left[\frac{d}{dx}\cosh(x)\right]}{\cosh^2(x)}$

"tanch"

$= \frac{\cosh^2(x) - \sinh^2(x)}{\cosh^2(x)} = \frac{1}{\cosh^2(x)} = \text{sech}^2(x)$

~~sech~~ "seech"

④

$14°\ \frac{d}{dx}\operatorname{sech}(x) = \frac{d}{dx}\left(\frac{1}{\cosh(x)}\right) = \frac{d}{dx}\left(\cosh(x)\right)^{-1}$

$= -1\cdot\left(\cosh(x)\right)^{-2}\cdot\frac{d}{dx}\cosh(x)$

$= \frac{-1}{\cosh^2(x)}\cdot\sinh(x) = -\frac{\sinh(x)}{\cosh(x)}\cdot\frac{1}{\cosh(x)}$

$= -\tanh(x)\operatorname{sech}(x)$

$15°\ \frac{d}{dx}\operatorname{arc}\cancel{\cosh}\tanh(x) = \frac{1}{1-x^2}$

$\tanh(\underbrace{\operatorname{arctanh}(x)}_{\tanh^{-1}(x)}) = x \Rightarrow \frac{d}{dx}\tanh(\operatorname{arctanh}(x)) = \frac{d}{dx}x = 1$

$= \operatorname{sech}^2(\operatorname{arctanh}(x))\cdot\frac{d}{dx}\operatorname{arctanh}(x)$

$= \left(1-\tanh^2(\operatorname{arctanh}(x))\right)\cdot\frac{d}{dx}\operatorname{arctanh}(x)$

$= \left(1-x^2\right)\frac{d}{dx}\operatorname{arctanh}(x)$

$\tanh^2(x) - 1$

$= \frac{\sinh^2(x)}{\cosh^2(x)} - \frac{\cosh^2(x)}{\cosh^2(x)}$

$= \frac{\sinh^2(x) - \cosh^2(x)}{\cosh^2(x)}$

$= \frac{-1}{\cosh^2(x)} = -\operatorname{sech}^2(x)$

$\therefore \operatorname{sech}^2(x) = 1 - \tanh^2(x)$