Derivatives III – The Chain Rule

6° Chain Rule

$$\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x)$$

$$\overset{\shortparallel}{(f \circ g)'(x)}$$

"f follows g of x"

es $\quad \frac{d}{dx} \sin(x^2) = \cos(x^2) \cdot \frac{d}{dx} x^2 = \cos(x^2) \cdot 2x$

[since $\frac{d}{dt} \sin(t) = \cos(t)$]

Why is it called the "Chain Rule"?

$y = f(u) \qquad u = g(x)$

$$\frac{dy}{dx} = \underset{\substack{\uparrow \\ f'(g(x))}}{\frac{dy}{du}} \cdot \underset{\substack{\uparrow \\ g'(x)}}{\frac{du}{dx}}$$

Why does it work?

$$(f\circ g)'(x) = \lim_{h\to 0} \frac{(f\circ g)(x+h) - (f\circ g)(x)}{h} = \lim_{h\to 0} \frac{f(g(x+h)) - f(g(x))}{h}$$

$$= \lim_{h\to 0} \frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)} \cdot \frac{g(x+h) - g(x)}{h}$$

$$= \left[\lim_{h\to 0} \frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)}\right] \overbrace{\left[\lim_{h\to 0} \frac{g(x+h) - g(x)}{h}\right]}^{=g'(x)}$$

$$= \left[\lim_{h\to 0} \frac{f(g(x) + (g(x+h) - g(x))) - f(g(x))}{g(x+h) - g(x)}\right] \cdot g'(x)$$

Change variable in this limit:
$t = g(x+h) - g(x)$ Note that $t \to 0$ as $h \to 0$
since $g(x+h) \to g(x)$ as $h \to 0$,
(because $g(x)$ is differentiable).

$$= \left[\lim_{t\to 0} \frac{f(g(x)+t) - f(g(x))}{t}\right] \cdot g'(x)$$

The derivative of f at $g(x)$.

$$= f'(g(x)) \cdot g'(x).$$

Examples: 1) $\frac{d}{dx}\sin^2(x) = \frac{d}{dx}(\sin(x))^2$

Chain Rule works from outside in, so handle the square first (Power Rule).

$= 2\sin(x)\cdot\frac{d}{dx}\sin(x)$

$= 2\sin(x)\cos(x)$ $[=\sin(2x)]$

2) $\frac{d}{dx}\tan(e^{\sqrt{x}}) \overset{\text{Chain Rule}}{=} \sec^2(e^{\sqrt{x}})\cdot\frac{d}{dx}e^{\sqrt{x}}$

$\overset{\text{Chain Rule again}}{=} \sec^2(e^{\sqrt{x}})\cdot e^{\sqrt{x}}\cdot\frac{d}{dx}\sqrt{x}$

$= \sec^2(e^{\sqrt{x}})\cdot e^{\sqrt{x}}\cdot\frac{d}{dx}x^{1/2}$

$\overset{\text{Power Rule}}{=} \sec^2(e^{\sqrt{x}})\cdot e^{\sqrt{x}}\cdot\frac{1}{2}x^{\frac{1}{2}-1}$

$= \sec^2(e^{\sqrt{x}})\cdot e^{\sqrt{x}}\cdot\frac{1}{2}x^{-1/2}$

$= \sec^2(e^{\sqrt{x}})\cdot e^{\sqrt{x}}\cdot\frac{1}{2\sqrt{x}}$

$$\boxed{\begin{aligned}
\frac{d}{dt}\tan(t) &= \frac{d}{dt}\left[\frac{\sin(t)}{\cos(t)}\right] \quad \text{Quotient Rule} \\
&= \frac{\left(\frac{d}{dt}\sin(t)\right)\cos(t) - \sin(t)\left(\frac{d}{dt}\cos(t)\right)}{(\cos(t))^2} \\
&= \frac{\cos(t)\cdot\cos(t) + \sin(t)[\cancel{-}\sin(t)]}{\cos^2(t)} \\
&= \frac{\cos^2(t)+\sin^2(t)}{\cos^2(t)} = \frac{1}{\cos^2(t)} \\
&= \left(\frac{1}{\cos(t)}\right)^2 = (\sec(t))^2 = \sec^2(t)
\end{aligned}}$$

3) e^x and $\ln(x)$ are each other's inverses

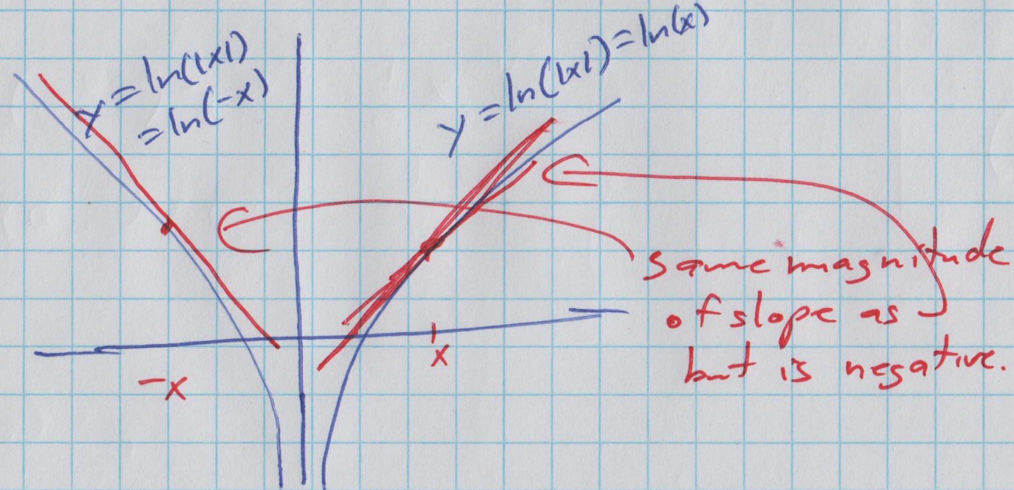
ie $\ln(e^x) = x$ and $e^{\ln(x)} = x$

$$1 = \frac{d}{dx}x = \frac{d}{dx}e^{\ln(x)} = e^{\ln(x)} \cdot \frac{d}{dx}\ln(x) \qquad \left[\frac{d}{dt}e^t = e^t\right]$$

$$= x \cdot \frac{d}{dx}\ln(x)$$

ie $1 = x\frac{d}{dx}\ln(x) \implies \frac{d}{dx}\ln(x) = \frac{1}{x}$ [For $x > 0$, since that's where $\ln(x)$ is defined.]

Aside: In fact, $\frac{d}{dx}\ln(|x|) = \frac{1}{x}$ (for all $x \neq 0$)



Proposition: Suppose $f(x)$ and $f^{-1}(x)$ are both differentiable.

$\{$So $f^{-1}(f(x)) = x$ and $f(f^{-1}(x)) = x.\}$

Then

$$1 = \frac{d}{dx} x = \frac{d}{dx} f(f^{-1}(x)) = f'(f^{-1}(x)) \cdot \frac{d}{dx} f^{-1}(x)$$

$$\therefore \frac{d}{dx} f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}$$

Example: $\arctan(x) = \tan^{-1}(x)$ is the inverse of $\tan(x)$

We know $\frac{d}{dx}\tan(x) = \sec^2(x)$. What is $\frac{d}{dx}\arctan(x)$?

$$1 = \frac{d}{dx} x = \frac{d}{dx}\tan(\arctan(x)) = \sec^2(\arctan(x)) \cdot \frac{d}{dx}\arctan(x)$$

or, using the Proposition,

$$\frac{d}{dx}\arctan(x) = \frac{1}{\sec^2(\arctan(x))} = \frac{1}{1+\tan^2(\arctan(x))}$$

$$= \frac{1}{1+[\tan(\arctan(x))]^2} = \frac{1}{1+[x]^2} = \frac{1}{1+x^2}$$

$$1+\tan^2(x)$$
$$= \frac{\cos^2(x)}{\cos^2(x)} + \frac{\sin^2(x)}{\cos^2(x)}$$
$$= \frac{\cos^2(x)+\sin^2(x)}{\cos^2(x)}$$
$$= \frac{1}{\cos^2(x)} = \sec^2(x)$$

Next time: More work to build up our library of derivatives.