Limits V - Two special limits and a theorem

1° $\lim_{\theta \to 0} \frac{\theta}{\sin(\theta)} = 1$

As $\theta \to 0$, $\sin(\theta) \to 0$...

so $\frac{\theta}{\sin(\theta)} \begin{matrix} \to 0 \\ \to 0 \end{matrix}$ is "indeterminate".

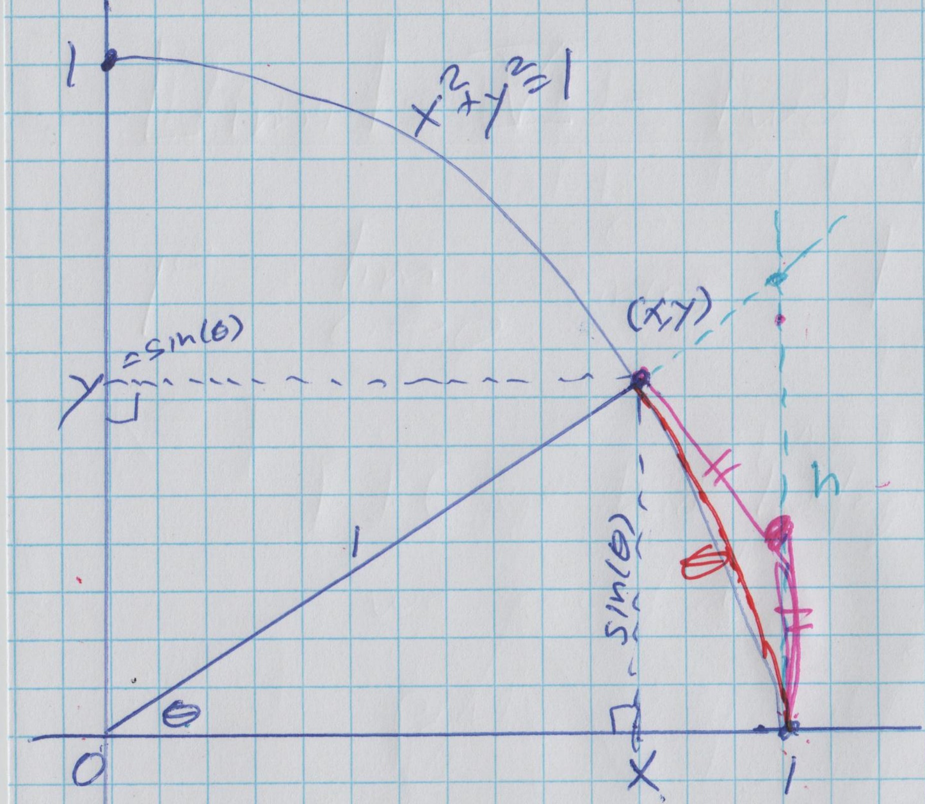
⌈If we had l'Hôpital's Rule and knew the derivative of sin(x),

we could do: $\lim_{\theta \to 0} \frac{\theta}{\sin(\theta)} \begin{matrix} \to 0 \\ \to 0 \end{matrix} = \lim_{\theta \to 0} \frac{\frac{d}{d\theta}\theta}{\frac{d}{d\theta}\sin(\theta)} = \lim_{\theta \to 0} \frac{1}{\cos(\theta)}$

$= \frac{1}{1} = 1$ since $\cos(\theta) \to 1$ as $\theta \to 0$. ⌋

Problem: We'll need derivatives (and don't yet have them) and we need something like this limit to get the derivative of sin(x).

We'll look at the geometry and see what we can do...



$$\sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{x}{1} = y$$

$$\cos(\theta) = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{x}{1} = x$$

Note: $\sin(\theta) \leq \theta$, so for $\theta > 0$

$$\frac{\theta}{\sin(\theta)} \geq 1.$$

Let h be the height of the larger triangle.

Observe that $\frac{h}{1} = \frac{y}{x} = \frac{\sin(\theta)}{\cos(\theta)}$ $[= \tan(\theta)]$.

Also, $\theta \leq h$. $[\theta \leq \neq + \neq \leq h]$

If we measure θ in radians and we have a unit circle then an angle of θ at the centre subtends an arc of length θ.

So we have

$$\sin(\theta) \leq \theta \leq \frac{\sin(\theta)}{\cos(\theta)}$$

$$\Rightarrow \quad 1 \leq \frac{\theta}{\sin(\theta)} \leq \frac{1}{\cos(\theta)}$$

(as long as $\sin(\theta) \neq 0$)

$$\boxed{\lim_{\theta \to 0} \frac{\theta}{\sin(\theta)} = 1}$$

As $\theta \to 0$, $1 \to 1$ and $\cos(\theta) \to 1$, so does $\frac{1}{\cos(\theta)} \to \frac{1}{1} = 1$. By the Squeeze Thm, $\frac{\theta}{\sin(\theta)} \to 1$ as well.

It follows that $\lim\limits_{\theta \to 0} \dfrac{\sin(\theta)}{\theta} = \dfrac{1}{\lim\limits_{\theta \to 0} \dfrac{\theta}{\sin(\theta)}} = \dfrac{1}{1} = 1.$

(We'll use this when we first work out the derivative of $\sin(x)$.)

$2°$ $\lim\limits_{h \to 0} \dfrac{e^h - 1}{h} \begin{matrix} \to 0 \\ \to 0 \end{matrix}$ but $= 1$

[We can't use l'Hôpital's Rule for the same reasons as before.]

From the latter part of MATH 1120H - Calculus II:

$$e^x = \frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots$$

$n! = n(n-1)(n-2)\cdots 2 \cdot 1$

$[0! = 1]$

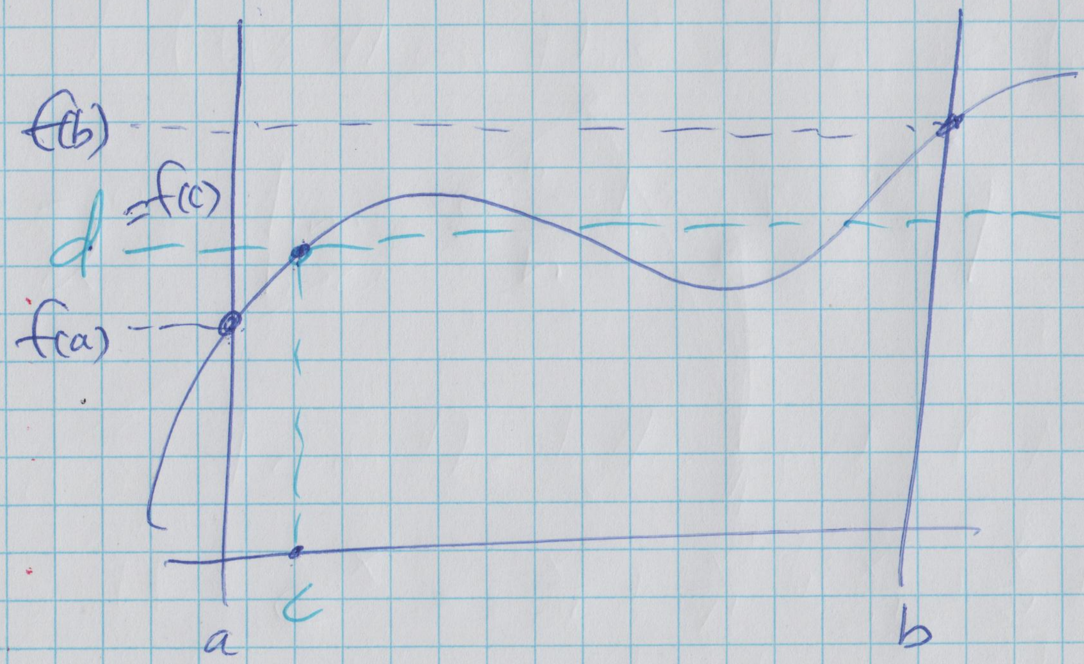
is the power series expansion of e^x [ie the Taylor series about 0 of e^x; ie the MacLaurin series of e^x]

$$\lim_{h \to 0} \frac{e^h - 1}{h} = \lim_{h \to 0} \frac{(\cancel{1} + h + \frac{h^2}{2} + \frac{h^3}{6} + \cdots) - \cancel{1}}{h}$$

$$= \lim_{h \to 0} \frac{h + \frac{h^2}{2} + \frac{h^3}{6} + \frac{h^4}{24} + \cdots}{h} = \lim_{h \to 0} \frac{\cancel{h}(1 + \overset{\to 0}{\frac{h}{2}} + \overset{\to 0}{\frac{h^2}{6}} + \overset{\to 0}{\cdots})}{\cancel{h}} = 1 + 0 + 0 + \cdots$$

$$= 1 \checkmark$$

3° Intermediate Value Theorem



If $f(x)$ is continuous on $[a,b]$ and d is a number between ~~and~~ $f(a)$ and $f(b)$, then there is a $c \in [a,b]$, ~~then the~~ such that $f(c) = d$.

We'll be using this implicitly (and sometimes explicitly) quite a bit later on.

<u>Next time</u>: We define derivatives!