Mathematics 1110H – Calculus I: Limits, derivatives, and Integrals
TRENT UNIVERSITY, Fall 2019

Assignment #2
Games With Limits
Due on Wednesday, 2 October.

The usual $\varepsilon - \delta$ definition of limits,

> DEFINITION. $\lim_{x \to a} f(x) = L$ exactly when for every $\varepsilon > 0$ there is a $\delta > 0$ such that for any x with $|x - a| < \delta$ we are guaranteed to have $|f(x) - L| < \varepsilon$ as well.

is pretty hard to wrap your head around the first time or three for most people. Here is less common definition, equivalent to the one above, which recasts the confusing logical structure of the above definition in terms of a game:

> ALTERNATE DEFINITION. The *limit game* for $f(x)$ at $x = a$ with target L is a three-move game played between two players A and B as follows:
>
> 1. A moves first, picking a small number $\varepsilon > 0$.
> 2. B moves second, picking another small number $\delta > 0$.
> 3. A moves third, picking an x that is within δ of a, *i.e.* $a - \delta < x < a + \delta$.
>
> To determine the winner, we evaluate $f(x)$. If it is within ε of the target L, *i.e.* $L - \varepsilon < f(x) < L + \varepsilon$, then player B wins; if not, then player A wins.
>
> With this idea in hand, $\lim_{x \to a} f(x) = L$ means that player B has a winning strategy in the limit game for $f(x)$ at $x = a$ with target L; that is, if B plays it right, B will win no matter what A tries to do. (Within the rules ... :-) Conversely, $\lim_{x \to a} f(x) \neq L$ means that player A is the one with a winning strategy in the limit game for $f(x)$ at $x = a$ with target L.

The game definition of limits isn't really better or worse that the usual $\varepsilon - \delta$ definition, but each is easier for some people to understand, and the exercise in trying it both ways usually helps in understanding what is really going on here.

1. Use one of these two definitions of limit to verify that $\lim_{x \to 1} (-2x + 1) = -1$. *[2]*

2. Use the definition of limit that you didn't use in answering question **1** to verify that $\lim_{x \to 2} (-x + 2) \neq 1$. *[2]*

3. Use either definition of limits above to verify that $\lim_{x \to 3} \left(x^2 - 5\right) = 2$. *[3]*

Hint: The choice of δ in **3** will probably require some slightly indirect reasoning. Pick some arbitrary smallish positive number for δ as a first cut. If it doesn't do the job, but x is at least that close, you'll have more information to help pin down the δ you really need.

NOTE: The problems above are probably easiest done by hand, though `Maple` and its competitors do have tools for solving inequalities which could be useful.

4. Use `Maple` (or a program with similar capabilities) to compute $\lim_{x \to 0} \dfrac{\sin(x + \pi)}{x}$. *[2]*

5. Compute $\lim_{x \to 0} \dfrac{\sin(x + \pi)}{x}$ by hand. *[1]*