Mathematics 1110H – Calculus I: Limits, derivatives, and Integrals
Trent University, Fall 2019

Assignment #4
Approximating Definite Integrals
Due on Wednesday, 20 November.

You might want to skim through Chapter 7 and Section 8.6 of the textbook for fuller (if still incomplete) discussions of definite integrals and numerical approximations of same.

The definite integral $\displaystyle\int_a^b f(x)\,dx$ represents the signed or weighted area of the region between $y = f(x)$ and the x-axis for $a \leq x \leq b$, where area above the x-axis is added and area below the x-axis is subtracted. The definite integral is usually defined in terms of limits of "Riemann sums", but the full general definition, while necessary to justify all the properties of definite integrals and to handle a pretty wide range of functions, is also pretty cumbersome to work with. For a lot of purposes, we can get by with a much simpler definition, such as the Right-Hand Rule, given below, which suffices to develop at least some of the properties of the definite integral and will, in principle, properly compute $\displaystyle\int_a^b f(x)\,dx$ for most commonly encountered functions. As a reminder:

Right-Hand Rule. Suppose $f(x)$ is defined for all x in $[a, b]$ and is continuous at all but finitely many points of $[a, b]$. Then:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \left[\sum_{i=1}^{n} \frac{b-a}{n} f\left(a + i\cdot\frac{b-a}{n}\right)\right] = \lim_{n\to\infty} \left[\frac{b-a}{n} \sum_{i=1}^{n} f\left(a + i\cdot\frac{b-a}{n}\right)\right]$$

The idea here is that we divide up the interval $[a, b]$ into n subintervals of equal width $\dfrac{b-a}{n}$, so the ith subinterval, going from left to right and where $1 \leq i \leq n$, will be $\left[(i-1)\cdot\dfrac{b-a}{n}, i\cdot\dfrac{b-a}{n}\right]$. Each subinterval serves as the base of a rectangle of height $f\left(a + i\cdot\dfrac{b-a}{n}\right)$, which must then have area $\dfrac{b-a}{n} f\left(a + i\cdot\dfrac{b-a}{n}\right)$. The sum of the areas of these rectangles, the *nth Right-Hand Rule sum for* $\displaystyle\int_a^b f(x)\,dx$, approximates the area computed by $\displaystyle\int_a^b f(x)\,dx$. (It's called the Right-Hand Rule because it uses the right-hand endpoint of each subinterval to evaluate $f(x)$ at to determine the height of the rectangle which has that subinterval as a base.) As we increase n and so shrink the width of the rectangles we get better and better approximations to the definite integral. The object of this assignment is to work out how quickly the approximations get better as long as the drivative of $f(x)$ is well-behaved.

1. Suppose $|f'(x)| \leq M$ for all $x \in [a, b]$, where $M \geq 0$ is a constant. Show that

$$\left|\int_a^b f(x)\,dx - \frac{b-a}{n} \sum_{i=1}^{n} f\left(a + i\cdot\frac{b-a}{n}\right)\right| \leq \frac{M(b-a)^2}{n} \qquad [5]$$

Hint: Show that the error contributed by the ith rectangle in the Right-Hand Rule sum is at most $\frac{M(b-a)^2}{n^2}$. To see how that might work, draw a picture of what is going on at the top of this rectangle. The discussion of the more sophisticated Trapezoid and Simpson's Rules in §8.6 of our textbook is a useful model here.

2. Using the formula given in **1**, how large would n have to be to guarantee that the nth Right-Hand Rule sum for $\int_{-2}^{2} \left(4 - x^2\right)\, dx$ is within 0.01 of the correct value of the definite integral. *[2]*

3. Use `Maple` to compute both $\int_{-2}^{2} \left(4 - x^2\right)\, dx$ and the nth Right-Hand Rule sum for this definite integral for the n you worked out in your answeer to **2**. Is the difference between them indeed at most $\frac{M(b-a)^2}{n^2}$, using the M and n from your solution to **2**? *[3]*

Hint: If using `Maple`'s worksheet mode, you'll want to look up the `int` and `sum` operators.