Trent University, Fall 2018

MATH 1110H Test

Friday, 2 November

Time: 11:00–11:50
Space: SC 137

Name: M.Y. Solutions

Student Number: 2.718281

Question	Mark
1	____
2	____
3	____
Total	____ /30

Instructions

- *Show all your work.* Legibly, please! Simplify where you reasonably can.
- *If you have a question, ask it!*
- Use the back sides of all the pages for rough work or extra space.
- You may use a calculator and (all sides of) an aid sheet.

1. Compute $\dfrac{dy}{dx}$ for any *four* (4) of parts **a**–**f**. *[12 = 4 × 3 each]*

a. $y = \ln(\sec(x) + \tan(x))$ **b.** $(x+y)^2 = x^2 + y^2 + 1$ **c.** $y = \dfrac{x^2+1}{x+2}$

d. $y = \cos(2x)\sin(2x)$ **e.** $y = \sinh(x) + \cosh(x)$ **f.** $y = e^{\sqrt{x}}$

SOLUTIONS. **a.** Chain Rule and trig derivatives, with some algebra to simplify afterwards.

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\ln(\sec(x)+\tan(x)) = \frac{1}{\sec(x)+\tan(x)} \cdot \frac{d}{dx}(\sec(x)+\tan x)\\ &= \frac{1}{\sec(x)+\tan(x)} \cdot \left(\sec(x)\tan(x)+\sec^2(x)\right) = \frac{\sec(x)\left(\tan(x)+\sec(x)\right)}{\sec(x)+\tan(x)} = \sec(x) \quad \blacksquare\end{aligned}$$

b. *[Solve for y first.]* Since $(x+y)^2 = x^2+2xy+y^2$, it follows from $(x+y)^2 = x^2+y^2+1$ that $2xy = 1$, so $y = \dfrac{1}{2x} = \dfrac{1}{2} \cdot x^{-1}$. This means, using the Power Rule, that:

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{1}{2} \cdot x^{-1}\right) = \frac{1}{2} \cdot (-1)x^{-2} = -\frac{1}{x^2} \quad \blacksquare$$

b. *[Differentiate first.]* Implicit differentiation, here we come! With a bit of the Chain and Power Rules, too:

$$\begin{aligned}&\frac{d}{dx}\left((x+y)^2\right) = \frac{d}{dx}\left(x^2+y^2+1\right) \implies 2(x+y)\frac{d}{dx}(x+y) = 2x + 2y\frac{dy}{dx} + 0\\ &\implies 2(x+y)\left(1+\frac{dy}{dx}\right) = 2x+2y\frac{dy}{dx} \implies 2(x+y) + 2(x+y)\frac{dy}{dx} = 2x + 2y\frac{dy}{dx}\\ &\implies 2(x+y)\frac{dy}{dx} - 2y\frac{dy}{dx} = 2x - 2(x+y) \implies 2x\frac{dy}{dx} = -2y \implies \frac{dy}{dx} = -\frac{2y}{2x} = -\frac{y}{x}\end{aligned}$$

That's as far as this goes without solving for y, but it's enough for full marks. ■

c. Quotient and Power Rules.

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\left(\frac{x^2+1}{x+2}\right) = \frac{\left[\frac{d}{dx}\left(x^2+1\right)\right](x+2) - \left(x^2+1\right)\left[\frac{d}{dx}(x+2)\right]}{(x+2)^2}\\ &= \frac{[2x](x+2) - \left(x^2+1\right)[1]}{(x+2)^2} = \frac{2x^2+4x-x^2-1}{(x+2)^2} = \frac{x^2+4x-1}{(x+2)^2} \quad \blacksquare\end{aligned}$$

d. *[Simplify first.]* We'll use the double angle formula for sin, followed by the Chain Rule:

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(\cos(2x)\sin(2x)) = \frac{d}{dx}\left(\frac{1}{2} \cdot 2\cos(2x)\sin(2x)\right) = \frac{d}{dx}\frac{1}{2}\sin(4x)\\ &= \frac{1}{2}\cos(4x) \cdot \frac{d}{dx}(4x) = \frac{1}{2}\cos(4x) \cdot 4 = 2\cos(4x) \quad \blacksquare\end{aligned}$$

d. *[Differentiate first.]* Product and Chain Rules, away! Double angle formula for cos, help out!

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\left(\cos(2x)\sin(2x)\right) = \left[\frac{d}{dx}\cos(2x)\right]\cdot\sin(2x) + \cos(2x)\cdot\left[\frac{d}{dx}\sin(2x)\right] \\ &= \left[-\sin(2x)\frac{d}{dx}(2x)\right]\cdot\sin(2x) + \cos(2x)\left[\cos(2x)\frac{d}{dx}(2x)\right] \\ &= \left[-2\sin(2x)\right]\cdot\sin(2x) + \cos(2x)\left[2\cos(2x)\right] \\ &= -2\sin^2(2x) + 2\cos^2(2x) = 2\cos(4x) \quad \blacksquare\end{aligned}$$

e. *[Differentiate away.]* $\dfrac{dy}{dx} = \dfrac{d}{dx}\left(\sinh(x)+\cosh(x)\right) = \cosh(x)+\sinh(x)$ If you would like to simplify this, see below. $\blacksquare$

e. *[Simplify (?) first.]* We'll go back to the definition of sinh and cosh and do a little algebra first:

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\left(\sinh(x)+\cosh(x)\right) = \frac{d}{dx}\left(\frac{e^x-e^{-x}}{2}+\frac{e^x+e^{-x}}{2}\right) \\ &= \frac{d}{dx}\left(\frac{e^x-e^{-x}+e^x+e^{-x}}{2}\right) = \frac{d}{dx}\left(\frac{2e^x}{2}\right) = \frac{d}{dx}e^x = e^x \quad \blacksquare\end{aligned}$$

f. Chain Rule and a bit of the Power Rule:

$$\frac{dy}{dx} = \frac{d}{dx}e^{\sqrt{x}} = e^{\sqrt{x}}\cdot\frac{d}{dx}\sqrt{x} = e^{\sqrt{x}}\cdot\frac{d}{dx}x^{1/2} = e^{\sqrt{x}}\cdot\frac{1}{2}x^{-1/2} = \frac{e^{\sqrt{x}}}{2\sqrt{x}} \quad \blacksquare$$

2. Do any *two* (2) of parts **a**–**e**. *[8 = 2 × 4 each]*

a. Compute $\displaystyle\lim_{t\to\infty} \frac{\sin(t)+\cos(t)}{t}$.

b. Find the maximum value of $f(x) = e^{-x^2}$ for $-2 \le x \le 2$.

c. Use the ε–δ definition of limits to verify that $\displaystyle\lim_{x\to -1}(3x+2) = -1$.

d. Find the equation of the tangent line to $y = \ln(x)$ at $x = 1$.

e. Use the limit definition of the derivative to verify that $\dfrac{d}{dx}x^3 = 3x^2$.

Solutions. **a.** This is a job for the Squeeze Theorem. Since $-1 \le \sin(t) \le 1$ and $-1 \le \cos(t) \le 1$ for all t, we have that $-2 \le \sin(t)+\cos(t) \le 2$ for all t, and hence that $-\dfrac{2}{t} \le \dfrac{\sin(t)+\cos(t)}{t} \le \dfrac{2}{t}$ for all $t > 0$. Since $\displaystyle\lim_{t\to\infty}\left(-\frac{2}{t}\right) = 0 = \lim_{t\to\infty}\frac{2}{t}$, it follows by the Squeeze Theorem that $\displaystyle\lim_{t\to\infty}\frac{\sin(t)+\cos(t)}{t} = 0$, too. ■

b. We need to compare the values of $f(x)$ at the endpoints of the given interval with its values at any critical points inside the interval, and then select the largest among these. First, $f(-2) = e^{-(-2)^2} = e^{-4} = \dfrac{1}{e^4} \approx 0.2776$ and $f(2) = e^{-2^2} = e^{-4} = \dfrac{1}{e^4} \approx 0.2776$.

Second, $f'(x) = \frac{d}{dx}e^{-x^2} = e^{-x^2}\frac{d}{dx}\left(-x^2\right) = -2xe^{-x^2}$. Since $e^{-x^2} > 0$ for all x, it follows that $f'(x) = 0$ exactly when $x = 0$. This critical point is inside the given interval $[-2, 2]$, and $f(0) = e^{-0^2} = e^0 = 1$.

Finally, since $1 > 0.2776$, it follows that the maximum value of $f(x) = e^{-x^2}$ for x with $-2 \le x \le 2$ is $f(0) = 1$. ■

c. To verify that $\displaystyle\lim_{x\to -1}(3x+2) = -1$ using the ε–δ definition of limits, we need to, for any $\varepsilon > 0$, find a $\delta > 0$ such that for all x with $|x-(-1)| < \delta$, we also have $|(3x+2)-(-1)| < \varepsilon$. As usual, we reverse-engineer the required δ from ε:

$$\begin{aligned}|(3x+2)-(-1)| < \varepsilon &\iff |3x+3| < \varepsilon \iff 3\,|x+1| < \varepsilon \\ &\iff |x+1| < \frac{\varepsilon}{3} \iff |x-(-1)| < \frac{\varepsilon}{3}\end{aligned}$$

Note that each step above is completely reversible. It follows that for any $\varepsilon > 0$, if we set $\delta = \dfrac{\varepsilon}{3}$, then whenever $|x-(-1)| < \delta = \dfrac{\varepsilon}{3}$, we get $|(3x+2)-(-1)| < \varepsilon$ as well, as required.

Thus $\displaystyle\lim_{x\to -1}(3x+2) = -1$ by the ε–δ definition of limits. ■

d. First, the tangent line to $y = \ln(x)$ at $x = 1$ passes through the point $(1, \ln(1)) = (1, 0)$. Second, the slope of the tangent line to $y = \ln(x)$ at $x = 1$ is:

$$m = \left.\frac{dy}{dx}\right|_{x=1} = \left.\frac{d}{dx}\ln(x)\right|_{x=0} = \left.\frac{1}{x}\right|_{x=1} = \frac{1}{1} = 1$$

The tangent line thus has slope 1, and hence an equation of the form $y = x + b$. To determine b we use the fact that the tangent line passes through the point $(1, 0)$, *i.e.* $0 = 1 + b$. It follows that $b = -1$, and so the equation of the tangent line to $y = \ln(x)$ at $x = 1$ is $y = x - 1$. ■

e. The limit definition of the derivative states that $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$. We apply this to $f(x) = x^3$ below:

$$\begin{aligned}
f'(x) = \frac{d}{dx}x^3 &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^3 - x^3}{h} \\
&= \lim_{h \to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h} = \lim_{h \to 0} \frac{3x^2h + 3xh^2 + h^3}{h} \\
&= \lim_{h \to 0} \frac{h\left(3x^2 + 3xh + h^2\right)}{h} = \lim_{h \to 0} \left(3x^2 + 3xh + h^2\right) \\
&= 3x^2 + 3x \cdot 0 + 0^2 = 3x^2 \quad \blacksquare
\end{aligned}$$

3. Find the domain and any and all intercepts, intervals of increase and decrease, maximum and minimum points, intervals of curvature, and inflection points of the function $h(x) = xe^{-x}$, and sketch its graph based on this information. *[10]*

SOLUTION. *i. (Domain)* Since xe^{-x} makes sense no matter what real number x is plugged into the expression, the domain of $h(x) = xe^{-x}$ is $(-\infty, \infty)$, also known as $\mathbb{R}$, or "all x in $\mathbb{R}$", or just "all x", or ...

ii. (Intercepts) $h(0) = 0e^{-0} = 0 \cdot 1 = 0$, so $h(x)$ has y-intercept 0. Since $e^{-x} > 0$ for all x, $h(x) = xe^{-x} = 0$ only when $x = 0$, so $h(x)$ has 0 as its only x-intercept. Note that the y-intercept aand x-intercept are the same point, namely the origin.

iii. (Asymptotes) [Note that these weren't asked for, but just for drill ...] Since $h(x)$ is defined and continuous for all x, it has no vertical asymptotes. It remains to check for horizontal asymptotes:

$$\lim_{x\to-\infty} xe^{-x} = \infty \quad \text{since } x \to -\infty \text{ and } e^{-x} \to +\infty \text{ as } x \to -\infty$$

$$\begin{aligned}\lim_{x\to+\infty} xe^{-x} &= \lim_{x\to+\infty} \frac{x \;\to +\infty}{e^x \;\to +\infty} \quad \text{so we can apply l'Hôpital's Rule} \\ &= \lim_{x\to+\infty} \frac{\frac{d}{dx}x}{\frac{d}{dx}e^x} = \lim_{x\to+\infty} \frac{1 \;\to 1}{e^x \;\to +\infty} = 0^+\end{aligned}$$

Thus $h(x)$ has a horizontal asymptote, namely $y = 0$, only in the positive direction.

iv. (Increase/decrease/maxima/minima) First, we compute the derivative of $h(x)$.

$$\begin{aligned}h'(x) &= \frac{d}{dx}\left(xe^{-x}\right) = \left[\frac{d}{dx}x\right]e^{-x} + x\left[\frac{d}{dx}e^{-x}\right] = 1e^{-x} + xe^{-x}\left[\frac{d}{dx}(-x)\right] \\ &= e^{-x} + xe^{-x}(-1) = (1-x)e^{-x}\end{aligned}$$

Since $e^{-x} > 0$ for all x, $h'(x)$ is positive, negative, or zero, exactly as $1 - x$ is positive, negative, or zero. Note that $1 - x > 0$ exactly when $x < 1$, $1 - x < 0$ exactly when $x > 1$, and $1 - x = 0$ exactly when $x = 1$. Thus $h'(x) = (1 - x)e^{-x}$ is positive exactly when $x < 1$, negative exactly when $x > 1$, and zero exactly when $x = 1$. We relate this to the behaviour of $h(x)$ in the usual table:

x	$(-\infty, 1)$	1	$(1, \infty)$
$h'(x)$	$+$	0	$-$
$h(x)$	$\uparrow$	max	$\downarrow$

In particular, since $h(x)$ is increasing before $x = 1$ and decreasing after $x = 1$, the sole critical point, $x = 1$, is a maximum. Note that $h(1) = 1e^{-1} = \frac{1}{e} \approx 0.3679$.

v. (Curvature/inflection) First, we compute the second derivative of $h(x)$.

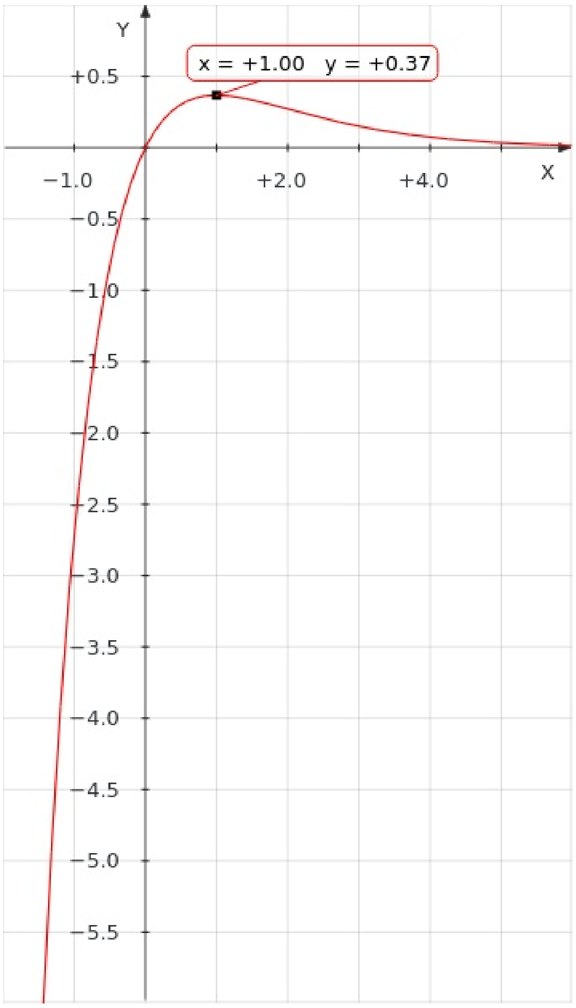
$$\begin{aligned}h''(x) &= \frac{d}{dx}h'(x) = \frac{d}{dx}(1-x)e^{-x} = \left[\frac{d}{dx}(1-x)\right]e^{-x} + (1-x)\left[\frac{d}{dx}e^{-x}\right] \\ &= [-1]e^{-x} + (1-x)e^{-x}\left[\frac{d}{dx}(-x)\right] = -e^{-x} + (1-x)e^{-x}[-1] \\ &= 1e^{-x} - e^{-x} + xe^{-x} = (x-2)e^{-x}\end{aligned}$$

Since $e^{-x} > 0$ for all x, $h''(x)$ is positive, negative, or zero, exactly as $x-2$ is positive, negative, or zero. Note that $x-2>0$ exactly when $x>2$, $x-2<0$ exactly when $x<2$, and $x-2=0$ exactly when $x=2$. Thus $h''(x)=(x-2)e^{-x}$ is positive exactly when $x>2$, negative exactly when $x<2$, and zero exactly when $x=2$. We relate this to the behaviour of $h(x)$ in the usual table:

x	$(-\infty, 2)$	2	$(2, \infty)$
$h''(x)$	$-$	0	$+$
$h(x)$	$\frown$	infl	$\smile$

Since $h(x)$ is concave to the left of $x=2$ and concave up to the right of $x=2$, it follows that $x=2$ is an inflection point of $h(x)$. Note that $h(2)=2e^{-2}=\frac{2}{e^2}\approx 0.2707$.

vi. (Graph) It's a cheat, but I had a computer draw the graph:



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