Mathematics 1110H – Calculus I: Limits, derivatives, and Integrals
TRENT UNIVERSITY, Summer 2018

Solutions to Assignment #2
Games With Limits

The usual definition of limits,

> $\varepsilon - \delta$ DEFINITION OF LIMITS. $\lim\limits_{x \to a} f(x) = L$ exactly when for every $\varepsilon > 0$ there is a $\delta > 0$ such that for any x with $|x - a| < \delta$ we are guaranteed to have $|f(x) - L| < \varepsilon$ as well.

is pretty hard to wrap your head around the first time or three for most people. Here is less common definition, which is still equivalent to the one above, that is cast in terms of a game:

> LIMIT GAME DEFINITION OF LIMITS. The *limit game* for $f(x)$ at $x = a$ with target L is a three-move game played between two players A and B as follows:
>
> 1. A moves first, picking a small number $\varepsilon > 0$.
> 2. B moves second, picking another small number $\delta > 0$.
> 3. A moves third, picking an x that is within δ of a, *i.e.* $a - \delta < x < a + \delta$.
>
> To determine the winner, we evaluate $f(x)$. If it is within ε of the target L, *i.e.* $L - \varepsilon < f(x) < L + \varepsilon$, then player B wins; if not, then player A wins.
>
> With this idea in hand, $\lim\limits_{x \to a} f(x) = L$ means that player B has a winning strategy in the limit game for $f(x)$ at $x = a$ with target L; that is, if B plays it right, B will win no matter what A tries to do. (Within the rules ... :-) Conversely, $\lim\limits_{x \to a} f(x) \neq L$ means that player A is the one with a winning strategy in the limit game for $f(x)$ at $x = a$ with target L.

Your task in this assignment, should you choose to accept it, is to find such winning strategies:

1. Describe a winning strategy for B in the limit game for $f(x) = 2x + 3$ at $x = 2$ with target 7. Note that no matter what number ε A plays first, B must have a way to find a δ to play that will make it impossible for A to play an x that wins for A on the third move. *[3]*

SOLUTION. For a winning strategy, B must, after A plays some $\varepsilon > 0$, have a way of finding a $\delta > 0$ such that no matter what x with $2 - \delta < x < 2 + \delta$ is played by A on the last move, B wins. B wins if $2x + 3$ is within ε of 7, so we try to reverse engineer a suitable δ from this condition:

$$\begin{aligned} 7 - \varepsilon < 2x + 3 < 7 + \varepsilon &\iff 4 - \varepsilon < 2x < 4 + \varepsilon \\ &\iff 2 - \frac{\varepsilon}{2} < x < 2 + \frac{\varepsilon}{2} \end{aligned}$$

This suggests that setting $\delta = \frac{\varepsilon}{2}$ will do as a winning strategy for B. It does:

Suppose A plays any $\varepsilon > 0$ and B responds with $\delta = \frac{\varepsilon}{2}$. No matter which particular x with $2 - \frac{\varepsilon}{2} < x < 2 + \frac{\varepsilon}{2}$ then gets played by A, we have that $4 - \varepsilon < 2x < 4 + \varepsilon$, and hence that $7 - \varepsilon < 2x + 3 < 7 + \varepsilon$, which is exactly what B needs to have to win. ■

2. Describe a winning strategy for A in the limit game for $f(x) = 2x + 3$ at $x = 2$ with target 8. Note that A must pick an ε on the first move so that no matter what δ B tries to play on the second move, A can still find an x to play on move three that wins for A. *[3]*

SOLUTION. A winning strategy for A has two parts. A must find – first part! – an $\varepsilon > 0$ such that no matter what $\delta > 0$ is then played by B, A has some way – second part! – of picking an x within δ of two for which $2x - 3$ is at least ε away from 8.

The trick to picking the $\varepsilon > 0$ is to check what the function $f(x) = 2x + 3$ is really doing near $x = 2$. Since linear functions are continuous, it shouldn't be a surprise that $2x + 3$ actually approaches 7, instead of 8, as x approaches 2. (Besides, we did problem **1**, didn't we? :-) Since the distance between 7 and 8 is 1, picking an $\varepsilon < 1$ should suffice to make it possible to find x's as close as you like to 2 which have $f(x) = 2x + 3$ at least ε away from 8. To keep it simple, let's use $\varepsilon = \frac{1}{2} = 0.5$. [First part of strategy complete!]

Now suppose B plays some $\delta > 0$ in response to $\varepsilon = 0.5$. $f(x) = 2x + 3$ is increasing, so any x to the left of 2 (*i.e.* with $x < 2$) will have $f(x) = 2x + 3 < 2 \cdot 2 + 3 = 7 < 8$, and hence have the distance between $f(x) = 2x + 3$ and 8 be greater than the distance between 7 and 8, which is 1, which in turn is greater than $\varepsilon = 0.5$. To be totally definite, we'll pick $x = 2 - \frac{\delta}{2}$. Then x is certainly within δ of 2, and the distance between $f(x)$ and 8 is:

$$8 - (2x + 3) = 5 - 2x = 5 - 2\left(2 - \frac{\delta}{2}\right) = 5 - 4 + \delta = 1 + \delta > 1 > 0.5 = \varepsilon$$

Since this works for any $\delta > 0$, it follows that A wins.

Thus picking $\varepsilon = 0.5 > 0$ and, once B plays a $\delta > 0$, playing $x = 2 - \frac{\delta}{2}$, is a winning strategy for A. ■

3. Use either definition of limits above to verify that $\lim_{x \to 2} \left(x^2 + 1\right) = 5$. *[4]*

Hint: The choice of δ in **3** will probably require some slightly indirect reasoning. Pick some arbitrary smallish positive number, say 1, for δ as a first cut. If it doesn't do the job, but x is at least that close, you'll have some more information to help pin down the δ you really need.

SOLUTION. We will use the game version of the $\varepsilon - \delta$ definition of limits. To verify that $\lim_{x \to 2} \left(x^2 + 1\right) = 5$, we need to find a winning strategy for playet B in the corresponding limit game. Player B wins if (s)he can always generate a δ in response to player A's ε such that no matter how A then picks an x with $2 - \delta < x < 2 + \delta$, we have $5 - \varepsilon < x^2 + 1 < 5 + \varepsilon$.

As usual we try to reverse-engineer a suitable δ from the winning condition, that is, from $5 - \varepsilon < x^2 + 1 < 5 + \varepsilon$.

$$\begin{aligned} 5 - \varepsilon < x^2 + 1 < 5 + \varepsilon &\iff -\varepsilon < x^2 - 4 < \varepsilon \iff -\varepsilon < (x-2)(x+2) < \varepsilon \\ &\iff -\frac{\varepsilon}{x+2} < x - 2 < \frac{\varepsilon}{x+2} \iff 2 - \frac{\varepsilon}{x+2} < x < 2 + \frac{\varepsilon}{x+2} \end{aligned}$$

Unfortunately, we can't use $\delta = \dfrac{\varepsilon}{x+2}$ because we do not know in advance what x player A will choose. (That choice happens after B plays a $\delta > 0$, after all.) However, we can get around this problem by somewhat limiting A's choice of x in advance by deciding that we will accept no $\delta > 1$, that is, decide in advance to require that $\delta \leq 1$. This means that, whatever $\delta > 0$ we may finally settle upon, once player A picks an x, we are guaranteed that:

$$\begin{aligned}2 - 1 \leq 2 - \delta < x < 2 + \delta \leq 2 + 1 &\Longrightarrow 1 < x < 3 \Longleftrightarrow 3 < x + 2 < 5 \\ &\Longleftrightarrow \frac{1}{3} > \frac{1}{x+2} > \frac{1}{5} \Longleftrightarrow \frac{\varepsilon}{3} > \frac{\varepsilon}{x+2} > \frac{\varepsilon}{5}\end{aligned}$$

Since $\delta \leq 1$ guarantees that $\frac{\varepsilon}{5} < \frac{\varepsilon}{x+2}$, if we also make sure that $\delta \leq \frac{\varepsilon}{5}$, then once player A picks any x with $2 - \delta < x < 2 + \delta$, we are guaranteed that $2 - \frac{\varepsilon}{x+2} < x < 2 + \frac{\varepsilon}{x+2}$ will be true. By our initial analysis, it then follows that $5 - \varepsilon < x^2 + 1 < 5 + \varepsilon$, which is exactly what player B needs to win.

A winning strategy for B, therefore, is to choose a $\delta > 0$ such that $\delta \leq 1$ and $\delta \leq \frac{\varepsilon}{5}$. If you wish to be more definite, setting δ to be the minimum of 1 and $\frac{\varepsilon}{5}$ will do. ■

NOTE: The problems above are probably easiest done by hand, though `Maple` and its competitors do have tools for solving inequalities which could be useful.