Trent University

MATH 1101Y Test 2

Monday, 3 February, 2014

Time: 50 minutes

Name: Solutions

Student Number: 3141592

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1	____
2	____
3	____
Total	____

Instructions

- *Show all your work.* Legibly, please!
- *If you have a question, ask it!*
- Use the extra page 4 and the back sides of all the pages for rough work or extra space.
- You may use a calculator and an aid sheet.

1. Compute any *four* (4) of the integrals in parts **u**–**z**. *[16 = 4 × 4 each]*

a. $\displaystyle\int \frac{1}{u\ln(u)}\,du$ **b.** $\displaystyle\int_0^{1/2} \cos(2\pi v)\,dv$ **c.** $\displaystyle\int w\sec^2(w)\,dw$

d. $\displaystyle\int_0^1 \left(x^2+1\right)^2\,dx$ **e.** $\displaystyle\int \sec^4(y)\,dy$ **f.** $\displaystyle\int_{-12}^{-9} \frac{1}{\sqrt{z+13}}\,dz$

Solutions. **a.** We will use the substitution $x = \ln(u)$, so $dx = \frac{1}{u}\,du$.

$$\int \frac{1}{u\ln(u)}\,du = \int \frac{1}{x}\,dx = \ln(x) + C = \ln\left(\ln(u)\right) + C \qquad \square$$

b. We will use the substitution $y = 2\pi v$, so $dy = 2\pi\,dv$ and thus $dv = \frac{1}{2\pi}\,dy$, and $\begin{smallmatrix} v & 0 & 1/2 \\ y & 0 & \pi \end{smallmatrix}$.

$$\begin{aligned}\int_0^{1/2} \cos(2\pi v)\,dv &= \int_0^{\pi} \cos(y)\,\frac{1}{2\pi}\,dy = \frac{1}{2\pi}\sin(y)\Big|_0^{\pi} \\ &= \frac{1}{2\pi}\sin(\pi) - \frac{1}{2\pi}\sin(0) = \frac{1}{2\pi}\cdot 0 - \frac{1}{2\pi}\cdot 0 = 0 \qquad \square\end{aligned}$$

c. We will use integration by parts with $u = w$ and $v' = \sec^2(w)$, so $u' = 1$ and $v = \tan(w)$.

$$\begin{aligned}\int w\sec^2(w)\,dw &= \int uv'\,dw = uv - \int u'v\,dw = w\tan(w) - \int 1\cdot\tan(w)\,dw \\ &= w\tan(w) - \ln\left(\sec(x)\right) + C \qquad \square\end{aligned}$$

d. We will use brute algebraic force.

$$\begin{aligned}\int_0^1 \left(x^2+1\right)^2\,dx &= \int_0^1 \left(x^4+2x^2+1\right)\,dx = \left(\frac{x^5}{5} + \frac{2x^3}{3} + x\right)\Big|_0^1 \\ &= \left(\frac{1^5}{5} + \frac{2\cdot 1^3}{3} + 1\right) - \left(\frac{0^5}{5} + \frac{2\cdot 0^3}{3} + 0\right) \\ &= \left(\frac{3}{15} + \frac{10}{15} + \frac{15}{15}\right) - 0 = \frac{28}{15} \qquad \square\end{aligned}$$

e. We will use the trigonometric identity $\sec^2(y) = 1 + \tan^2(y)$ and the substitution $u = \tan(y)$, so $du = \sec^2(y)\,dy$.

$$\begin{aligned}\int \sec^4(y)\,dy &= \int \sec^2(y)\sec^2(y)\,dy = \int \left(1+\tan^2(y)\right)\sec^2(y)\,dy = \int \left(1+u^2\right)\,du \\ &= u + \frac{u^3}{3} + C = \frac{1}{3}\tan^3(x) + \tan(x) + C \qquad \square\end{aligned}$$

f. We will use the substitution $w = z + 13$, so $dw = dz$ and $\begin{smallmatrix} z & -12 & -9 \\ w & 1 & 4 \end{smallmatrix}$.

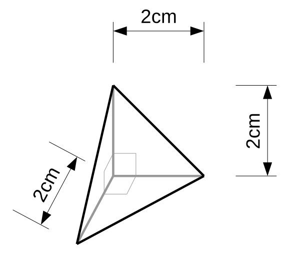
$$\begin{aligned}\int_{-12}^{-9} \frac{1}{\sqrt{z+13}}\,dz &= \int_1^4 \frac{1}{\sqrt{w}}\,dw = \int_1^4 w^{-1/2}\,dw = \frac{w^{1/2}}{1/2}\Big|_1^4 \\ &= 2\cdot 4^{1/2} - 2\cdot 1^{1/2} = 2\cdot 2 - 2\cdot 1 = 4 - 2 = 2 \qquad \blacksquare\end{aligned}$$

2. Do any *two* (2) of parts **g**–**k**. *[14 = 2 × 7 each]*

g. The volume of a solid obtained by revolving a region about the y-axis is given by $V = \int_0^{\pi/4} \pi \tan^2(y)\, dy$ when computed using the disk method. Sketch this solid and find its volume.

h. Find the average value of $f(x) = \cos(x) + \sin(x)$ on the interval $[0, \pi]$.

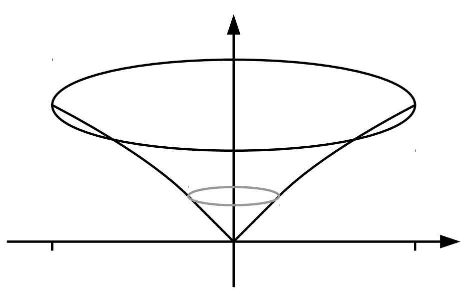
i. Compute $\int_0^2 x^2\, dx$ using the Left- or the Right-hand Rule.



j. A corner of a cube with sides 2 *cm* long is cut off along a plane passing through the three neighbouring vertices. Find the volume of this solid. [Picture at right! $\Longrightarrow$]

k. Sketch the region enclosed by $y = x^2$ and $x = y^2$ and find its area.

SOLUTIONS. **g.** First, a small sanity check: if one uses the disk/washer method and revolves around the y-axis, one should use y as the variable, which is the case in the given integral. Second, the volume formula for the disk/washer method in terms of y is $V = \int_a^b \pi\left(R^2 - r^2\right) dy$, where R and r are the outer and inner radius of the washer, respectively. Comparing this to the given integral, $V = \int_0^{\pi/4} \pi \tan^2(y)\, dy$, suggests that $R = \tan(y)$ and $r = 0$. It would follow that the region being revolved is the one to the left of $x = \tan(y)$ and to the right of $x = 0$, for $0 \le y \le \frac{\pi}{4}$. [In terms of x, this is the region below $y = \frac{\pi}{4}$ and above $y = \arctan(x)$, for $0 \le x \le 1$.] With this, we can sketch the region and the solid. Here is a crude sketch of the solid:



To compute the volume, we evaluate the integral, with the help of the trigonometric identity $\tan^2(y) = \sec^2(y) - 1$:

$$V = \int_0^{\pi/4} \pi \tan^2(y)\, dy = \pi \int_0^{\pi/4} \left(\sec^2(y) - 1\right) dy = \pi\left(\tan(y) - y\right)\Big|_0^{\pi/4}$$
$$= \pi\left(\tan\left(\frac{\pi}{4}\right) - \frac{\pi}{4}\right) - \pi\left(\tan(0) - 0\right) = \pi\left(1 - \frac{\pi}{4}\right) - \pi \cdot 0 = \pi - \frac{\pi^2}{4} \qquad \square$$

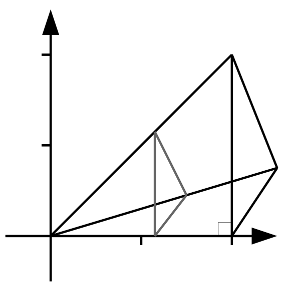
h. Recall that the average value of $f(x)$ on $[a, b]$ is given by $\dfrac{1}{b-a}\displaystyle\int_a^b f(x)\,dx$. Thus the average value of $f(x) = \cos(x) + \sin(x)$ on the interval $[0, \pi]$ is

$$\begin{aligned}\frac{1}{\pi - 0}\int_0^{\pi} (\cos(x) + \sin(x))\,dx &= \frac{1}{\pi}(\sin(x) - \cos(x))\Big|_0^{\pi} \\ &= \frac{1}{\pi}(\sin(\pi) - \cos(\pi)) - \frac{1}{\pi}(\sin(0) - \cos(0)) \\ &= \frac{1}{\pi}(0 - (-1)) - \frac{1}{\pi}(0 - 1) = \frac{1}{\pi} + \frac{1}{\pi} = \frac{2}{\pi}. \qquad \square\end{aligned}$$

i. We'll compute it using the Right-hand Rule; using the Left-hand Rule is pretty similar. Recall that the Right-hand Rule formula is $\displaystyle\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} \frac{b-a}{n} f\left(a + i\frac{b-a}{n}\right)$. In this case, $a = 0$, $b = 2$, and $f(x) = x^2$. Here goes:

$$\begin{aligned}\int_0^2 x^2\,dx &= \lim_{n\to\infty}\sum_{i=1}^{n}\frac{2-0}{n} f\left(0 + i\frac{2-0}{n}\right) = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{2}{n}\left(\frac{2i}{n}\right)^2 = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{2}{n}\cdot\frac{4}{n^2}\cdot i^2 \\ &= \lim_{n\to\infty}\frac{8}{n^3}\sum_{i=1}^{n} i^2 = \lim_{n\to\infty}\frac{8}{n^3}\cdot\frac{n(n+1)(2n+1)}{6} = \frac{8}{6}\lim_{n\to\infty}\frac{2n^2+3n+1}{n^2} \\ &= \frac{4}{3}\lim_{n\to\infty}\left(2 + \frac{3}{n} + \frac{1}{n^2}\right) = \frac{4}{3}(2+0+0) = \frac{8}{3} \qquad \square\end{aligned}$$

j. The trick here is to find some nice way to describe the areas of suitable cross-sections of the solid. We will do this by placing the solid so that one of the vertices the cutting plane went through is at the origin and the corner vertex is at $x = 2$, as in the diagram below.



The cross-section at x which is perpendicular to the x-axis is then a right triangle with base x and height x. (Think similar triangles if you can't see it ...) Thus the area of this cross section is $A(x) = \frac{1}{2}x^2$, and we also have that $0 \le x \le 2$. Thus the volume of the solid is given by:

$$V = \int_0^2 A(x)\,dx = \int_0^2 \frac{1}{2}x^2\,dx = \frac{1}{2}\cdot\frac{x^3}{3}\Big|_0^2 = \frac{1}{6}2^3 - \frac{1}{6}0^3 = \frac{8}{6} = \frac{4}{3} \qquad \square$$

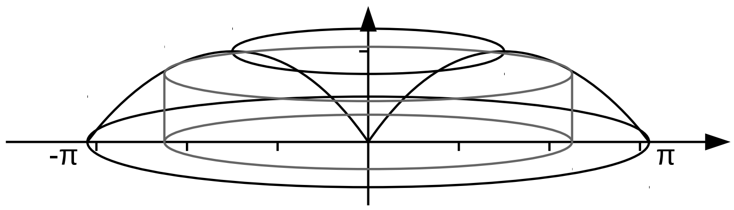
Part **k** is on page 4.

3. Do *one* (1) of parts **l** or **m**. Both involve the region between $y = \sin(x)$ and $y = 0$, where $0 \leq x \leq \pi$.

l. Sketch the solid of revolution obtained by revolving the given region about the y-axis and find its volume using the cylindrical shell method. *[10]*

m. Sketch the solid of revolution obtained by revolving the given region about the x-axis and find its volume using the disk/washer method. *[10]*

SOLUTIONS. **l.** Here is a sketch of the solid, with a cylindrical shell drawn in, too:



Since we revolved the region about the y-axis and are using shells, we should use x as the variable. Note that the shell at x has radius $R = x - 0 = x$ and height $h = \sin(x) - 0 = \sin(x)$. Thus the volume of the solid is given by:

$$V = \int_0^\pi 2\pi Rh\, dx = 2\pi \int_0^\pi x \sin(x)\, dx$$

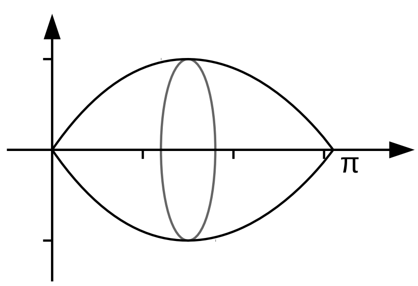
We'll use integration by parts with $u = x$ and $v' = \sin(x)$, so $u' = 1$ and $v = -\cos(x)$.

$$= 2\pi \left[-x\cos(x)\Big|_0^\pi - \int_0^\pi 1 \cdot (-\cos(x))\, dx \right]$$

$$= 2\pi \left[(-\pi\cos(\pi)) - (-0\cos(0)) + \int_0^\pi \cos(x)\, dx \right]$$

$$= 2\pi \left[-\pi(-1) - 0 + \sin(x)\big|_0^\pi \right] = 2\pi\left[\pi + \sin(\pi) - \sin(0)\right] = 2\pi\left[\pi + 0 + 0\right] = 2\pi^2 \qquad \square$$

m. Here is a sketch of the solid:



Since we revolved the region about the x-axis and are using disks/washers, we should use x as the variable. Note that the washer at x has outer radius $R = \sin(x) - 0 = \sin(x)$ and inner radius $r = 0 - 0 = 0$, so it is a disk. Thus the volume of the solid is given by:

4 m continued:

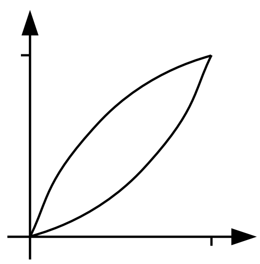
We'll use the trigonometric identity $\sin^2(x) = \frac{1}{2}(1 - \cos(2x))$, followed by the substitution $u = 2x$, so $du = 2dx$ and $dx = \frac{1}{2}\,du$, and $\begin{array}{ccc} x & 0 & \pi \\ u & 0 & 2\pi \end{array}$.

$$V = \int_0^\pi \pi R^2\,dx = \pi \int_0^\pi \sin^2(x)\,dx$$

$$= \pi \int_0^\pi \frac{1}{2}(1 - \cos(2x))\,dx = \frac{\pi}{2} \int_0^{2\pi} (1 - \cos(u))\,\frac{1}{2}\,du = \frac{\pi}{4}(u - \sin(u))\Big|_0^{2\pi}$$

$$= \frac{\pi}{4}(2\pi - \sin(2\pi)) - \frac{\pi}{4}(0 - \sin(0)) = \frac{\pi}{4}(2\pi - 0) - \frac{\pi}{4}(0 - 0) = \frac{\pi^2}{2} \quad \blacksquare$$

3 k. Note that $y = x^2$ and $x = y^2$ intersect when $x = \left(x^2\right)^2 = x^4$, *i.e.* when $x = 0$ or when $x^3 = 1$, which last occurs only when $x = 1$. Here is a sketch of the region:



The upper border is a piece of $x = y^2$, *i.e.* $y = \sqrt{x}$, and the lower border is a piece of $y = x^2$, for $0 \le x \le 1$ in both cases. The area is then given by:

$$A = \int_0^1 \left(\sqrt{x} - x^2\right)\,dx = \int_0^1 \left(x^{1/2} - x^2\right)\,dx = \left(\frac{x^{3/2}}{3/2} - \frac{x^3}{3}\right)\Big|_0^1$$

$$= \left(\frac{2}{3}1^{3/2} - \frac{1^3}{3}\right) - \left(\frac{2}{3}0^{3/2} - \frac{0^3}{3}\right) = \frac{1}{3} - 0 = \frac{1}{3} \quad \blacksquare$$

[Total = 40]

Bonus. Suppose a number of circles are drawn on a piece of paper, dividing it up into regions whose borders are made up of circular arcs. Show that one can always colour these regions using only black and white so that no two regions that have a border arc in common have the same colour. *[1]*

Solution. Colour each region white if it is inside an even number of circles, and black if it is inside an odd number of circles. (As in the given diagram!) If two regions share a border, then they are inside/outside all the same circles except for the one forming the border between them. One of the regions must be inside this circle and the other must be outside it. This means one of the regions is inside n circles for some n while the other is inside $n + 1$ circles. Since exactly one of n and $n + 1$ is even and the other is odd, the two regions get different colours, as required. $\blacksquare$