Trent University

MATH 1101Y Test #1

Monday, 11 November, 2013

Time: 50 minutes

Name: ______________________________

Student Number: ______________________________

Question	Mark
1	____
2	____
3	____
4	____
Total	____ /40

Instructions

- *Show all your work.* Legibly, please!
- *If you have a question, ask it!*
- Use the back sides of the test sheets for rough work or extra space.
- You may use a calculator and an aid sheet.

1. Find $\dfrac{dy}{dx}$ as best you can in any *three* (3) of **a**–**e**. *[12 = 3 × 4 each]*

a. $y = x\tan(x)$ **b.** $y = \dfrac{e^x}{x}$ **c.** $1 = \ln(xy)$ **d.** $y = \sin^3(x+41)$ **e.** $y = \dfrac{1}{1+\sqrt{x}}$

Solutions. **a.** Product Rule:

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\left(x\tan(x)\right) = \left(\frac{d}{dx}x\right)\tan(x) + x\left(\frac{d}{dx}\tan(x)\right)\\ &= 1\cdot\tan(x) + x\cdot\sec^2(x) = \tan(x) + x\sec^2(x) \qquad \square\end{aligned}$$

b. Quotient Rule:

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{e^x}{x}\right) = \frac{\left(\frac{d}{dx}e^x\right)x - e^x\left(\frac{d}{dx}x\right)}{x^2} = \frac{e^x\cdot x - e^x\cdot 1}{x^2} = \frac{(x-1)e^x}{x^2} \qquad \square$$

c. *Method i.* Solve for y first: $\ln(xy) = 1 \implies xy = e^{\ln(xy)} = e^1 = e \implies y = \dfrac{e}{x}$.
Then, using the Power Rule: $\dfrac{dy}{dx} = \dfrac{d}{dx}\left(\dfrac{e}{x}\right) = \dfrac{d}{dx}ex^{-1} = e(-1)x^{-2} = -\dfrac{e}{x^2}$. $\square$

Method ii. Implicit differentiation using the Chain and Product Rules:

$$0 = \frac{d}{dx}1 = \frac{d}{dx}\ln(xy) = \frac{1}{xy}\cdot\frac{d}{dx}(xy) = \frac{1}{xy}\left[\left(\frac{dx}{dx}\right)y + x\left(\frac{dy}{dx}\right)\right] = \frac{1}{xy}\left[1\cdot y + x\cdot\frac{dy}{dx}\right]$$

$$\implies \quad y + x\frac{dy}{dx} = xy\cdot 0 = 0 \quad \implies \quad \frac{dy}{dx} = -\frac{y}{x} \qquad \square$$

d. Power and Chain Rules:

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\sin^3(x+41) = 3\sin^2(x+41)\cdot\frac{d}{dx}\sin(x+41)\\ &= 3\sin^2(x+41)\cdot\cos(x+41)\cdot\frac{d}{dx}(x+41)\\ &= 3\sin^2(x+41)\cos(x+41)\cdot(1+0) = 3\sin^2(x+41)\cos(x+41) \qquad \square\end{aligned}$$

e. *Method i.* Power and Chain Rules:

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\left(\frac{1}{1+\sqrt{x}}\right) = \frac{d}{dx}\left(1+\sqrt{x}\right)^{-1} = (-1)\left(1+\sqrt{x}\right)^{-2}\cdot\frac{d}{dx}\left(1+\sqrt{x}\right)\\ &= -\left(1+\sqrt{x}\right)^{-2}\cdot\frac{d}{dx}\left(1+x^{1/2}\right) = -\left(1+\sqrt{x}\right)^{-2}\cdot\frac{1}{2}x^{-1/2} = \frac{-1}{2\sqrt{x}\left(1+\sqrt{x}\right)^2} \qquad \square\end{aligned}$$

Method ii. Quotient and Power Rules:

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\left(\frac{1}{1+\sqrt{x}}\right) = \frac{\left(\frac{d}{dx}1\right)\cdot\left(1+\sqrt{x}\right) - 1\cdot\frac{d}{dx}\left(1+\sqrt{x}\right)}{\left(1+\sqrt{x}\right)^2}\\ &= \frac{0\cdot\left(1+\sqrt{x}\right) - 1\cdot\left(0+\frac{1}{2\sqrt{x}}\right)}{\left(1+\sqrt{x}\right)^2} = \frac{-1}{2\sqrt{x}\left(1+\sqrt{x}\right)^2} \qquad \blacksquare\end{aligned}$$

2. Do any *two* (2) of **a**–**d**. *[10 = 2 × 5 each]*

a. Find the intercepts and the coordinates of the vertex of the parabola $y = x^2 - 2x - 3$.

b. Compute $\lim_{x \to 0} \frac{x^2}{\sin(x)}$.

c. Find $f^{-1}(x)$ for $f(x) = \frac{1}{1 + \sqrt{x}}$.

d. Use the limit definition of the derivative to find $f'(1)$ if $f(x) = x^2 + x$.

SOLUTIONS. **a.** When $x = 0$, $y = 0^2 - 2 \cdot 0 - 3 = -3$, so the y-intercept is at -3. Since $y = x^2 - 2x - 3 = (x+1)(x-3)$, which $= 0$ when $x = -1$ and when $x = 3$, the x-intercepts are at -1 and 3. (One could also find them using the quadratic formula.) The vertex of the parabola will be halfway between the x-intercepts, at $x = 1$, for which $y = 1^2 - 2 \cdot 1 - 3 = -4$, so the vertex is $(1, -4)$. (One could also find the vertex by completing the square.) □

b. *Method i.* Using algebra and the Limit Laws:

$$\lim_{x \to 0} \frac{x^2}{\sin(x)} = \lim_{x \to 0} \frac{x}{\frac{\sin(x)}{x}} = \frac{\lim_{x \to 0} x}{\lim_{x \to 0} \frac{\sin(x)}{x}} = \frac{0}{1} = 0 \qquad \square$$

Method ii. Using l'Hôpital's Rule, which is applicable since both $x^2 \to 0$ and $\sin(x) \to 0$ as $x \to 0$:

$$\lim_{x \to 0} \frac{x^2}{\sin(x)} = \lim_{x \to 0} \frac{\frac{d}{dx} x^2}{\frac{d}{dx} \sin(x)} = \lim_{x \to 0} \frac{2x}{\cos(x)} = \frac{2 \cdot 0}{\cos(0)} = \frac{0}{1} = 0 \qquad \square$$

c. As usual, we set $x = f(y)$ and try to solve for y:

$$x = f(y) = \frac{1}{1 + \sqrt{y}} \implies x\,(1 + \sqrt{y}) = 1 \implies 1 + \sqrt{y} = \frac{1}{x}$$
$$\implies \sqrt{y} = \frac{1}{x} - 1 \implies f^{-1}(x) = y = \left(\frac{1}{x} - 1\right)^2$$

Note that $f(x) = \frac{1}{1+\sqrt{x}}$ has domain $[0, \infty)$ and range $(0, 1]$, while $f^{-1}(x) = \left(\frac{1}{x} - 1\right)^2$ has domain $x \neq 0$ and range $[0, \infty)$. You can amuse yourself working out that asymmetry. □

d. Here goes:

$$f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0} \frac{\left[(1+h)^2 + (1+h)\right] - \left[1^2 + 1\right]}{h}$$
$$= \lim_{h \to 0} \frac{\left[1^2 + 2 \cdot 1 \cdot h + h^2 + 1 + h\right] - 2}{h} = \lim_{h \to 0} \frac{3h + h^2}{h}$$
$$= \lim_{h \to 0} \frac{h(3+h)}{h} = \lim_{h \to 0} (3 + h) = 3 + 0 = 3 \qquad \blacksquare$$

3. Do *one* (1) of **a** or **b**. *[8]*

a. A Borg cube's volume expands proportionately to how much matter it ingests: every 100 kg of matter ingested adds 1 m^3 to the volume. If the Borg cube ingests matter at a constant rate of 3000 kg/s, how quickly is each side of the cube growing at the instant that each side of the cube measures 10 m?

b. What is the maximum area of a rectangle whose total perimeter is 16 m?

SOLUTIONS. **a.** A cube with side length s has volume $V = s^3$. We are told that the volume expands at a (constant!) rate of

$$\frac{dV}{dt} = 3000\ kg/s \cdot \frac{1}{100}\ m^3/kg = 30\ m^3/s\,.$$

On the other hand,

$$\frac{dV}{dt} = \frac{d}{dt}s^3 = \left(\frac{d}{ds}s^3\right) \cdot \frac{ds}{dt} = 3s^2\frac{ds}{dt}\,.$$

Combining these and solving for $\frac{ds}{dt}$, we get

$$\frac{ds}{dt} = \frac{\frac{dV}{dt}}{3s^2} = \frac{30}{3s^2} = \frac{10}{s^2}\,.$$

We want to know what $\frac{ds}{dt}$ is when $s = 10\ m$:

$$\left.\frac{ds}{dt}\right|_{s=10} = \frac{10}{10^2} = \frac{1}{10} = 0.1\ m/s \qquad \square$$

b. Suppose the rectangle has height h and width w; its perimeter is then $P = 2h+2w = 16$ and its area is $A = hw$. The former equation implies that $2h = 16 - 2w$, *i.e.* $h = 8 - w$, so

$$A = (8 - w)w = 8w - w^2\,.$$

Note that $0 \le w \le 8$; at $w = 0$ all the perimeter of the rectangle is concentrated in the height, and at $w = 8$, all the perimeter of the rectangle is concentrated in the width. At either extreme, the area of the rectangle is 0.

Taking the derivative,

$$\frac{dA}{dw} = \frac{d}{dw}\left(8w - w^2\right) = 8 - 2w\,.$$

This $= 0$ exactly when $w = \frac{8}{2} = 4$, at which point $A = (8 - 4)4 = 4^2 = 16$. This must be a maximum since the area is 0 at the endpoints $w = 0$ and $w = 8$.

Thus the maximum area of the rectangle whose total perimeter is 16 m is 16 m^2. ■

4. Find the domain and all the intercepts, vertical and horizontal asymptotes, maxima and minima, and points of inflection of $f(x) = \dfrac{x^2+1}{x}$, and sketch its graph. *[10]*

SOLUTION. We run through the checklist:

i. Domain. $f(x) = \frac{x^2+1}{x}$ is a rational function, which is defined (and continuous and differentiable) unless the denominator is 0, so the domain consists of all $x \neq 0$.

ii. Intercepts. Since $f(x)$ is undefined at $x = 0$, the function has no y-intercept.

Since $y = f(x) = \frac{x^2+1}{x} = 0$ only if $x^2 + 1 = 0$, and $x^2 + 1 \geq 1 > 0$ for all x, $f(x)$ cannot equal 0, so there are no x-intercepts either.

iii. Vertical asymptotes. Since $f(x) = \frac{x^2+1}{x}$ is defined and continuous for all $x \neq 0$, the only place there might be a vertical asymptote is at 0. We check in the usual way:

$$\lim_{x \to 0^-} \frac{x^2+1}{x} = \lim_{x \to 0^-} \left(x + \frac{1}{x}\right) = 0^- + (-\infty) = -\infty$$
$$\lim_{x \to 0^+} \frac{x^2+1}{x} = \lim_{x \to 0^+} \left(x + \frac{1}{x}\right) = 0^+ + (+\infty) = +\infty$$

Thus the function has a vertical asymptote at 0, going down to $-\infty$ on the left and up to $+\infty$ on the right.

iv. Horizontal asymptotes. We check in the usual way:

$$\lim_{x \to -\infty} \frac{x^2+1}{x} = \lim_{x \to -\infty} \left(x + \frac{1}{x}\right) = -\infty + 0 = -\infty$$
$$\lim_{x \to +\infty} \frac{x^2+1}{x} = \lim_{x \to +\infty} \left(x + \frac{1}{x}\right) = +\infty + 0 = +\infty$$

Thus the function has no horizontal asymptotes.

v. Increase. decrease, maxima, and minima. Derivatives at last!

$$f'(x) = \frac{d}{dx}\left(\frac{x^2+1}{x}\right) = \frac{d}{dx}\left(x + \frac{1}{x}\right) = 1 - \frac{1}{x^2} = \frac{x^2-1}{x^2}$$

It follows that $f'(x) = 0$ exactly when $x^2 - 1 = 0$, *i.e.* exactly when $x = \pm 1$. Since $x^2 > 0$ for all $x \neq 0$, $f'(x) = \frac{x^2-1}{x^2}$ is positive or negative exactly when $x^2 - 1$ is positive or negative. $x^2 - 1 > 0$ exactly when $x^2 > 1$, *i.e.* when $x < -1$ or when $x > 1$, and $x^2 - 1 < 0$ exactly when $x^2 < 1$, *i.e.* when $-1 < x < 1$. Thus $f(x)$ is increasing when $x < -1$ or $x > 1$, and decreasing when $-1 < x < 1$, so it has a (local) maximum at $x = -1$ and a (local) minimum at $x = 1$. We summarize all of this in the usual table, recalling that $f(x)$ is undefined at $x = 0$ (as is $f'(x)$, too):

x	$(-\infty, -1)$	-1	$(-1, 0)$	0	$(0, 1)$	1	$(1, \infty)$
$f'(x)$	$+$	0	$-$	undef.	$-$	0	$+$
$f(x)$	$\uparrow$	max	$\downarrow$	undef.	$\downarrow$	min	$\uparrow$

Note that $f(-1) = \frac{(-1)^2+1}{-1} = -2$ and $f(1) = \frac{1^2+1}{1} = 2$. Since the minimum we found is larger than the maximum we found, they are only local, and not absolute, extreme points.

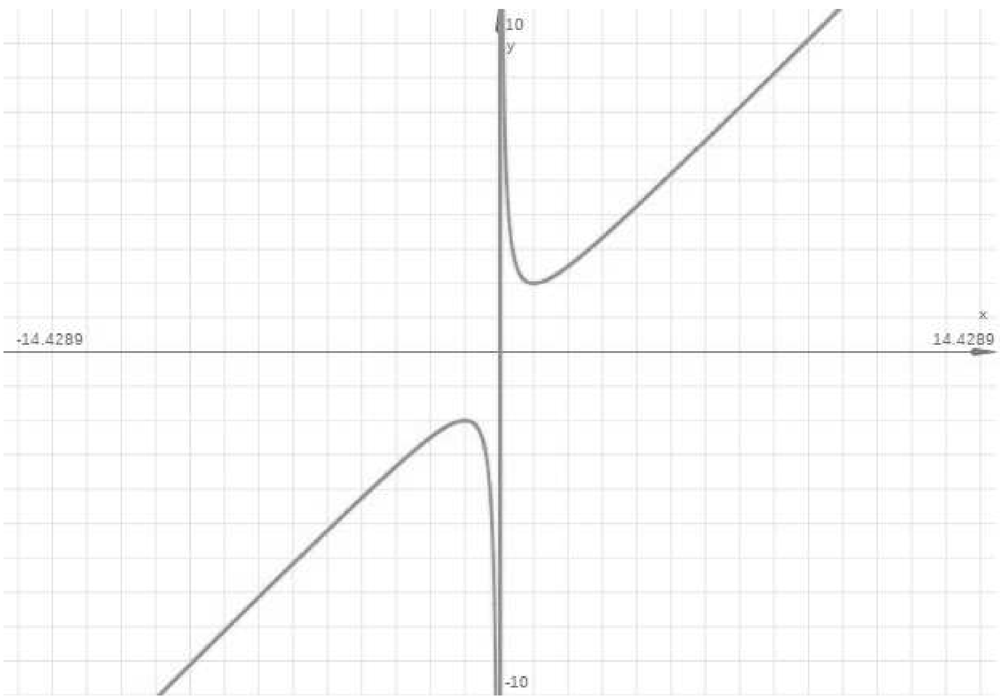
vi. Concavity and points of inflection. More derivatives!

$$f''(x) = \frac{d}{dx} f'(x) = \frac{d}{dx}\left(\frac{x^2-1}{x^2}\right) = \frac{d}{dx}\left(1 - \frac{1}{x^2}\right) = 0 - \left(\frac{-2}{x^3}\right) = \frac{2}{x^3}$$

It follows that $f''(x)$ is never equal to 0. It is, however, undefined for $x = 0$, and when $x < 0$, $f''(x) < 0$, while when $x > 0$, $f''(x) > 0$, so $f(x)$ is concave down for $x < 0$ and concave up for $x > 0$. (Since $f(x)$ is undefined at $x = 0$, it does not actually have an inflection point there.) We summarize all of this in another table:

x	$(-\infty, 0)$	0	$(0, \infty)$
$f''(x)$	$-$	undef.	$+$
$f(x)$	$\frown$	undef.	$\smile$

vii. The graph. Cheating slightly, I used `KAlgebra` to plot the graph:



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