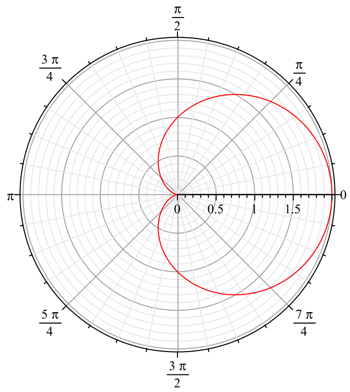
Mathematics 1101Y – Calculus I: Functions and calculus of one variable
TRENT UNIVERSITY, 2013–2014

Solutions to Assignment #♡
♡ Curves

A *cardioid* is one of a family of heart-shaped curves; the polar curve $r = 1 + \cos(\theta)$, for $0 \le \theta \le 2\pi$, from problem **4** on Assignment #1 is a common example of one:



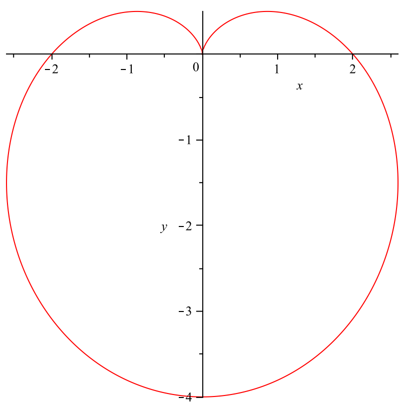
In this assignment we will consider the very similar cardioid given

- algebraically in Cartesian coordinates by $\left(x^2+y^2+2y\right)^2 = 4\left(x^2+y^2\right)$;
- parametrically in Cartesian coordinates by $x = \dfrac{8t}{\left(t^2+1\right)^2}$ and $y = \dfrac{4\left(t^2-1\right)}{\left(t^2+1\right)^2}$, for $-\infty < t < \infty$ [technically, this parametrization omits the point $(0,0)$]; and
- in polar coordinates by $r = 2\left(1-\sin(\theta)\right)$, for $0 \le \theta \le 2\pi$.

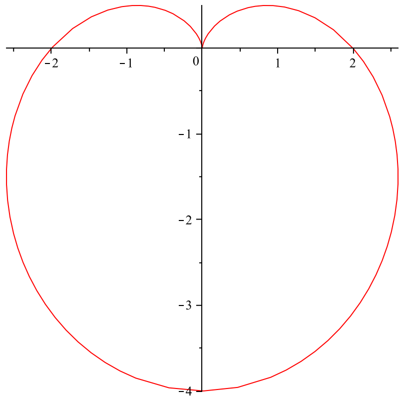
1. Plot all three descriptions of the given cardioid. *[1.5]*

SOLUTION.

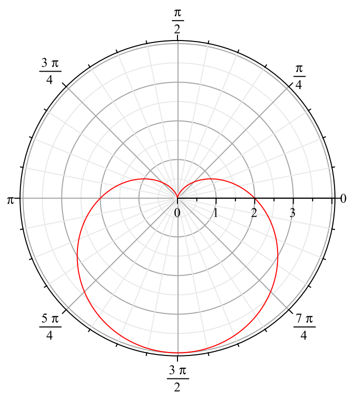
```
> with(plots)
> implicitplot( (x^2+y^2+2*y)^2=4*(x^2+y^2), x=-4..4, y =-5..1, gridrefine=4 )
```



```
> plot([8*t/(t^2+1)^2, (4*(t^2-1))/(t^2+1)^2, t=-100..100])
```




```
> polarplot(2*(1-sin(t)), t=0..2*Pi)
```



■

2. Pick two of the three descriptions and show that all the points given by one of them are also given by the other. *[2.5]*

SOLUTION. We'll show that the equation $\left(x^2+y^2+2y\right)^2 = 4\left(x^2+y^2\right)$ and the parametric curve $x = \dfrac{8t}{\left(t^2+1\right)^2}$ and $y = \dfrac{4\left(t^2-1\right)}{\left(t^2+1\right)^2}$, for $-\infty < t < \infty$, give the same points [except for $(0,0)$, which the parametric curve omits].

First, suppose $t \in (-\infty, \infty)$, and $x = \dfrac{8t}{\left(t^2+1\right)^2}$ and $y = \dfrac{4\left(t^2-1\right)}{\left(t^2+1\right)^2}$, We need to check that $\left(x^2+y^2+2y\right)^2 = 4\left(x^2+y^2\right)$. On the simpler hand,

$$\begin{aligned}
4\left(x^2+y^2\right) &= 4\left(\left[\frac{8t}{\left(t^2+1\right)^2}\right]^2 + \left[\frac{4\left(t^2-1\right)}{\left(t^2+1\right)^2}\right]^2\right) = 4\left(\frac{64t^2}{\left(t^2+1\right)^4} + \frac{16\left(t^4-2t^2+1\right)}{\left(t^2+1\right)^4}\right) \\
&= 4\frac{64t^2+16t^4-32t^2+16}{\left(t^2+1\right)^4} = 4\frac{16t^4+32t^2+16}{\left(t^2+1\right)^4} = \frac{64t^4+128t^2+64}{\left(t^2+1\right)^4} \\
&= \frac{64\left(t^4+2t^2+1\right)}{\left(t^2+1\right)^4} = \frac{64\left(t^2+1\right)^2}{\left(t^2+1\right)^4} = \frac{64}{\left(t^2+1\right)^2},
\end{aligned}$$

and on the more complicated hand, reusing the part of the above calculation in which we worked out $x^2 + y^2$ in terms of t, we have

$$
\begin{aligned}
\left(x^2+y^2+2y\right)^2 &= \left(\left[\frac{8t}{\left(t^2+1\right)^2}\right]^2 + \left[\frac{4\left(t^2-1\right)}{\left(t^2+1\right)^2}\right]^2 + 2\frac{4\left(t^2-1\right)}{\left(t^2+1\right)^2}\right)^2 \\
&= \left(\frac{16t^4+32t^2+16}{\left(t^2+1\right)^4} + \frac{8\left(t^2-1\right)}{\left(t^2+1\right)^2}\right)^2 = \left(\frac{16\left(t^4+2t^2+1\right)}{\left(t^2+1\right)^4} + \frac{8t^2-8}{\left(t^2+1\right)^2}\right)^2 \\
&= \left(\frac{16\left(t^2+1\right)^2}{\left(t^2+1\right)^4} + \frac{8t^2-8}{\left(t^2+1\right)^2}\right)^2 = \left(\frac{16}{\left(t^2+1\right)^2} + \frac{8t^2-8}{\left(t^2+1\right)^2}\right)^2 \\
&= \left(\frac{8t^2+8}{\left(t^2+1\right)^2}\right)^2 = \left(\frac{8\left(t^2+1\right)}{\left(t^2+1\right)^2}\right)^2 = \left(\frac{8}{t^2+1}\right)^2 = \frac{64}{\left(t^2+1\right)^2} = 4\left(x^2+y^2\right) .
\end{aligned}
$$

Thus every point on the parametric curve $x = \dfrac{8t}{\left(t^2+1\right)^2}$ and $y = \dfrac{4\left(t^2-1\right)}{\left(t^2+1\right)^2}$, for $-\infty < t < \infty$, is on the Cartesian curve $\left(x^2+y^2+2y\right)^2 = 4\left(x^2+y^2\right)$.

Second, suppose (x, y) is a point other than $(0, 0)$ [which is omitted by the parametrization] on the curve $\left(x^2+y^2+2y\right)^2 = 4\left(x^2+y^2\right)$. We need to show that there is a $t \in (-\infty, \infty)$ such that $x = \dfrac{8t}{\left(t^2+1\right)^2}$ and $y = \dfrac{4\left(t^2-1\right)}{\left(t^2+1\right)^2}$. If $(x, y) = (0, -4)$, then it is easy to see that $t = 0$ does the job. Since we don't have to worry about the points $(0, 0)$ and $(0, -4)$, we may suppose that $x \neq 0$. We can then reverse-engineer the required t. Rearranging $x = \dfrac{8t}{\left(t^2+1\right)^2}$ and $y = \dfrac{4\left(t^2-1\right)}{\left(t^2+1\right)^2}$ gives us $\left(t^2+1\right)^2 = \dfrac{8t}{x}$ and $y\left(t^2+1\right)^2 = 4\left(t^2-1\right)$, so $\dfrac{8ty}{x} = 4\left(t^2-1\right)$ and hence $8yt = 4xt^2 - 4x$. Rearranging this a little gives us $4xt^2 - 8yt - 4x = 0$, and dividing by 4 makes this $xt^2 - 2yt - x = 0$. Applying the quadratic formula now tells us that

$$
t = \frac{-(-2y) \pm \sqrt{(-2y)^2 - 4 \cdot x \cdot (-x)}}{2x} = \frac{2y \pm \sqrt{4y^2+4x^2}}{2x} = \frac{2y \pm 2\sqrt{x^2+y^2}}{2x} = \frac{y \pm \sqrt{x^2+y^2}}{x} .
$$

Since there is always a t that does the job – you figure out which of the two possibilities it might be! – every point on the curve $\left(x^2+y^2+2y\right)^2 = 4\left(x^2+y^2\right)$ [except $(0, 0)$] is on the parametric curve $x = \dfrac{8t}{\left(t^2+1\right)^2}$ and $y = \dfrac{4\left(t^2-1\right)}{\left(t^2+1\right)^2}$, for $-\infty < t < \infty$. ■

3. Find the area of the region enclosed by the given cardioid. *[3]*

Hint: This is most easily done in polar coordinates. You can look up how to compute areas in polar coordinates in §11.4.

SOLUTION. We apply the area formula $A = \int_a^b \frac{1}{2} r^2 \, d\theta$ to the curve $r = 2\left(1 - \sin(\theta)\right)$, for $0 \leq \theta \leq 2\pi$, and integrate away:

$$A = \int_a^b \frac{1}{2} r^2\, d\theta = \int_0^{2\pi} \frac{1}{2} \left[2\left(1-\sin(\theta)\right)\right]^2\, d\theta = \frac{4}{2}\int_0^{2\pi} \left(1 - 2\sin(\theta) + \sin^2(\theta)\right)\, d\theta$$
$$= \frac{1}{2}\int_0^{2\pi} 1\, d\theta - \frac{2}{2}\int_0^{2\pi} \sin(\theta)\, d\theta + \frac{1}{2}\int_0^{2\pi} \sin^2(\theta)\, d\theta$$
$$= \frac{1}{2}\theta\Big|_0^{2\pi} - \left(-\cos(\theta)\right)\Big|_0^{2\pi} + \frac{1}{2}\int_0^{2\pi} \frac{1}{2}\left(1-\cos(2\theta)\right)\, d\theta \qquad \begin{array}{l}\text{Use } u = 2\theta, \text{ so } du = 2d\theta,\\ \textit{i.e. } d\theta = \frac{1}{2}du \text{ and } \begin{smallmatrix}\theta & 0 & 2\pi\\ u & 0 & \pi\end{smallmatrix}.\end{array}$$
$$= \frac{1}{2}(2\pi - 0) + \cos(\theta)\Big|_0^{2\pi} + \frac{1}{4}\int_0^{\pi} \left(1-\cos(u)\right)\frac{1}{2}\, du = \pi + \cos(2\pi) - \cos(0) + \frac{1}{8}\left(u - \sin(u)\right)\Big|_0^{\pi}$$
$$= \pi + 1 - 1 + \frac{1}{8}\left(\pi - \sin(\pi)\right) - \frac{1}{8}\left(0 - \sin(0)\right) = \pi + \frac{1}{8}(\pi - 0) - \frac{1}{8}(0-0) = \frac{9}{8}\pi \qquad \blacksquare$$

4. Find the arc-length of the given cardioid. *[3]*

Hint: You can look up how to compute the arc-length of a curve in the textbook, too: §8.1 for doing so for Cartesian curves, §11.2 for parametric curves, and §11.4 for polar curves.

SOLUTION. This is also easiest using the polar form of the curve. We apply the arc-length formula for polar curves, arc-length $= \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$, to the curve $r = 2\left(1-\sin(\theta)\right)$ (so $\frac{dr}{d\theta} = -2\cos(\theta)$), for $0 \le \theta \le 2\pi$, and integrate away, overcoming all obstacles as we encounter them:

$$\text{arc-length} = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = \int_0^{2\pi} \sqrt{\left[2\left(1-\sin(\theta)\right)\right]^2 + \left[-2\cos(\theta)\right]^2}\, d\theta$$
$$= \int_0^{2\pi} \sqrt{4 - 8\sin(\theta) + 4\sin^2(\theta) + 4\cos^2(\theta)}\, d\theta \qquad \text{but } \sin^2(\theta) + \cos^2(\theta) = 1, \text{ so}$$
$$= \int_0^{2\pi} \sqrt{8 - 8\sin(\theta)}\, d\theta = \int_0^{2\pi} \sqrt{8\left(1-\sin(\theta)\right)}\, d\theta = 2\sqrt{2}\int_0^{2\pi} \sqrt{1-\sin(\theta)}\, d\theta$$
$$= 2\sqrt{2}\int_0^{2\pi} \sqrt{1-\sin(\theta)} \cdot \frac{\sqrt{1+\sin(\theta)}}{\sqrt{1+\sin(\theta)}}\, d\theta = 2\sqrt{2}\int_0^{2\pi} \frac{\sqrt{\left(1-\sin(\theta)\right)\left(1+\sin(\theta)\right)}}{\sqrt{1+\sin(\theta)}}\, d\theta$$
$$= 2\sqrt{2}\int_0^{2\pi} \frac{\sqrt{1-\sin^2(\theta)}}{\sqrt{1+\sin(\theta)}}\, d\theta = 2\sqrt{2}\int_0^{2\pi} \frac{\sqrt{\cos^2(\theta)}}{\sqrt{1+\sin(\theta)}}\, d\theta = 2\sqrt{2}\int_0^{2\pi} \frac{|\cos(\theta)|}{\sqrt{1+\sin(\theta)}}\, d\theta$$

and, since $\cos(\theta)$ isn't always equal to $|\cos(\theta)|$ for $0 \le \theta \le 2\pi$, we split the integral up

$$= 2\sqrt{2}\int_0^{\pi/2} \frac{\cos(\theta)}{\sqrt{1+\sin(\theta)}}\, d\theta + 2\sqrt{2}\int_{\pi/2}^{3\pi/2} \frac{-\cos(\theta)}{\sqrt{1+\sin(\theta)}}\, d\theta + 2\sqrt{2}\int_{3\pi/2}^{2\pi} \frac{\cos(\theta)}{\sqrt{1+\sin(\theta)}}\, d\theta$$

and we substitute $u = 1 + \sin(\theta)$, so $du = \cos(\theta)\, d\theta$ and $\begin{smallmatrix}\theta & 0 & \pi/2 & 3\pi/2 & 2\pi\\ u & 1 & 2 & 0 & 1\end{smallmatrix}$ in each part,

$$= 2\sqrt{2}\int_1^2 \frac{1}{\sqrt{u}}\, du + 2\sqrt{2}\int_2^0 \frac{-1}{\sqrt{u}}\, du + 2\sqrt{2}\int_0^1 \frac{1}{\sqrt{u}}\, du$$
$$= 2\sqrt{2}\int_1^2 \frac{1}{\sqrt{u}}\, du + 2\sqrt{2}\int_0^2 \frac{1}{\sqrt{u}}\, du + 2\sqrt{2}\int_0^1 \frac{1}{\sqrt{u}}\, du$$
$$= 2\sqrt{2} \cdot 2\sqrt{u}\Big|_1^2 + 2\sqrt{2} \cdot 2\sqrt{u}\Big|_0^2 + 2\sqrt{2} \cdot 2\sqrt{u}\Big|_0^1$$
$$= 2\sqrt{2} \cdot 2\sqrt{2} - 2\sqrt{2} \cdot 2\sqrt{1} + 2\sqrt{2} \cdot 2\sqrt{2} - 2\sqrt{2} \cdot 2\sqrt{0} + 2\sqrt{2} \cdot 2\sqrt{1} - 2\sqrt{2} \cdot 2\sqrt{0}$$
$$= 8 - 4\sqrt{2} + 8 - 0 + 4\sqrt{2} - 0 = 16$$

Whew! $\blacksquare$