Mathematics 1101Y – Calculus I: Functions and calculus of one variable
Trent University, 2013–2014

Solutions to Assignment #5
Epsilonics

One of the things we've skipped over was the formal definition of limit, that is, how to pin down just what $\lim_{x \to a} f(x) = L$ really means. The usual definition of limits is something like:

> $\varepsilon - \delta$ definition of limits. $\lim_{x \to a} f(x) = L$ exactly when for every $\varepsilon > 0$ there is a $\delta > 0$ such that for any x with $|x - a| < \delta$ we are guaranteed to have $|f(x) - L| < \varepsilon$ as well.

Informally, this means that no matter how close – that's the ε – you want $f(x)$ to get to L, you can make it happen by ensuring that x is close enough – that's the δ – to a. If this can always be done, $\lim_{x \to a} f(x) = L$; if not, then $\lim_{x \to a} f(x) \neq L$.

This definition works, but most people find it a little hard to understand and use at first. Here is less common definition equivalent to the one above that is cast in terms of a game:

> Limit game definition of limits. The *limit game* for $f(x)$ at $x = a$ with target L is a three-move game played between two players A and B as follows:
>
> 1. A moves first, picking a small number $\varepsilon > 0$.
> 2. B moves second, picking another small number $\delta > 0$.
> 3. A moves third, picking an x that is within δ of a, *i.e.* $a - \delta < x < a + \delta$.
>
> To determine the winner, we evaluate $f(x)$. If it is within ε of the target L, *i.e.* $L - \varepsilon < f(x) < L + \varepsilon$, then player B wins; if not, then player A wins.
>
> With this idea in hand, $\lim_{x \to a} f(x) = L$ means that player B has a winning strategy in the limit game for $f(x)$ at $x = a$ with target L; that is, if B plays it right, B will win no matter what A tries to do. (Within the rules ... :-) Conversely, $\lim_{x \to a} f(x) \neq L$ means that player A is the one with a winning strategy in the limit game for $f(x)$ at $x = a$ with target L.

Your task in this assignment, should you choose to accept it, is to find such winning strategies:

1. Describe a winning strategy for B in the limit game for $f(x) = 2x - 1$ at $x = 2$ with target 3. Note that no matter what number ε A plays first, B must have a way to find a δ to play that will make it impossible for A to play an x that wins for A on the third move. *[3]*

Solution. Whatever $\varepsilon > 0$ A may play, B will win by responding with $\delta = \frac{\varepsilon}{2}$. (Any positive δ which is even smaller will also work.) No matter what x A chooses with $|x - 2| < \delta = \frac{\varepsilon}{2}$, we have

$$|f(x) - 3| = |(2x - 1) - 3| = |2x - 4| = 2|x - 2| < 2\delta = 2\frac{\varepsilon}{2} = \varepsilon \,,$$

so B wins. ∎

Note: One gets $\delta = \varepsilon/2$ by reverse-engineering the δ from the desired conclusion, $|f(x) - 3| < \varepsilon$:

$$|f(x) - 3| < \varepsilon \iff |(2x - 1) - 3| < \varepsilon \iff |2x - 4| < \varepsilon \iff 2|x - 2| < \varepsilon \iff |x - 2| < \frac{\varepsilon}{2}$$

2. Describe a winning strategy for A in the limit game for $f(x) = 2x - 3$ at $x = 2$ with target 2. Note that A must pick an ε on the first move so that no matter what δ B tries to play on the second move, A can still find an x to play on move three that wins for A. *[3]*

SOLUTION. For the first move, let A play $\varepsilon = \frac{1}{2}$. (Any positive $\varepsilon \leq 1$ will also work.) No matter what $\delta > 0$ B plays in response, A can respond in turn with any x such that $2 < x < 2 + \delta$. Since

$$x > 2 \implies f(x) = 2x - 1 > 2 \cdot 2 - 1 = 3 > 2.5 = 2 + \frac{1}{2} = 2 + \varepsilon\,,$$

so $|f(x) - 2| = f(x) - 2 \geq \frac{1}{2} = \varepsilon$, which means that A wins. ■

NOTE: How does one figure out what ε to pick to begin with? You need one that is small enough to separate the target, 2, from where the function is really going, $f(2) = 3$. That is, any ε that is less than the distance between these two numbers, $|2 - 3| = 1$, will do. [Why does $\varepsilon = 1$ still – barely – do the job?]

3. Use either definition of limits above to verify that $\lim_{x \to 1} \left(x^2 + 2\right) = 3$. *[4]*

Hint: The choice of δ in **3** will probably require some slightly indirect reasoning. Pick some arbitrary smallish positive number, say 1, for δ as a first cut. If it doesn't do the job, but x is at least that close, you'll have some more information to help pin down the δ you really need.

SOLUTION. We'll use the standard ε–δ definition. As in the note after the solution to problem **1**, we will attempt to reverse-engineer the $\delta > 0$ required:

$$\begin{aligned} |f(x) - L| < \varepsilon \;\Leftrightarrow\; \left|\left(x^2 + 2\right) - 3\right| < \varepsilon \;\Leftrightarrow\; \left|x^2 - 1\right| < \varepsilon \\ \Leftrightarrow\; |(x+1)(x-1)| < \varepsilon \;\Leftrightarrow\; |x - 1| < \frac{\varepsilon}{|x+1|} \end{aligned}$$

The problem is that δ is not allowed to depend on x. (Recall that A plays x *after* B plays δ in the limit game definition.) Following the hint, we get around this problem by accepting no δ greater than 1. (Any number that is less than the distance between $x = 1$ and $x = -1$ will do, actually.) This lets us put bounds around $|x + 1|$ and so replace it with a constant in $\frac{\varepsilon}{|x+1|}$ above. If $|x - 1| < \delta \leq 1$, then

$$\begin{aligned} -1 < x - 1 < 1 \;&\Leftrightarrow\; 0 = -1 + 1 < x = x - 1 + 1 < 1 + 1 = 2 \\ &\Leftrightarrow\; 1 = 0 + 1 < x + 1 < 2 + 1 = 3 \\ &\Leftrightarrow\; 1 = \frac{1}{1} > \frac{1}{x+1} > \frac{1}{3}\,. \end{aligned}$$

Note that it also follows that if $|x - 1| < \delta \leq 1$, then $x + 1 > 0$ and so $|x + 1| = x + 1$. Thus, if $|x - 1| < \delta \leq 1$, we have $|x - 1| < \frac{\varepsilon}{3} < \frac{\varepsilon}{|x+1|}$.

It follows that if we let $\delta = \min\left(1, \frac{\varepsilon}{3}\right)$, the lesser of 1 and $\frac{\varepsilon}{3}$, then no matter what x is chosen that satisfies $|x - 1| < \delta$, we get that $\delta \leq 15$, and so

$$\begin{aligned} |x - 1| < \min\left(1, \frac{\varepsilon}{3}\right) \leq \frac{\varepsilon}{3} < \frac{\varepsilon}{|x+1|} \;&\Rightarrow\; |(x+1)(x-1)| < \varepsilon \\ &\Rightarrow\; \left|x^2 - 1\right| < \varepsilon \\ &\Rightarrow\; \left|\left(x^2 + 2\right) - 3\right| < \varepsilon\,, \end{aligned}$$

as the standard ε–δ definition requires to have that $\lim_{x \to 1} \left(x^2 + x + 1\right) = 3$. ■