Mathematics 1101Y – Calculus I: Functions and calculus of one variable
TRENT UNIVERSITY, 2013–2014
Final Examination

Time: *19:00–22:00, on Tuesday, 15 April, 2014.* *Brought to you by* Стефан Біланюк.

Instructions: Do parts **L**, **M**, and **N**, and, if you wish, part **O**. Show all your work and justify all your answers. *If in doubt about something,* **ask!**

Aids: Any calculator; (all sides of) one aid sheet; one (1) brain (no neuron limit).

Part L. Do all four (4) of **1–4**.

1. Compute $\frac{dy}{dx}$ as best you can in any *three* (3) of **a–f**. *[15 = 3 × 5 each]*

a. $y = \left(\frac{x+1}{x-1}\right)^2$ **b.** $y = \int_0^x te^{t^2}\, dt$ **c.** $\begin{array}{l} y = -\cos(t) \\ x = \sin(t) \end{array}$

d. $\ln(xy) = 0$ **e.** $y = \sin\left(\sqrt{x}\right)$ **f.** $y = x^\pi e^x$

2. Evaluate any *three* (3) of the integrals **a–f**. *[15 = 3 × 5 each]*

a. $\int \frac{e^{\sqrt{t}}}{2\sqrt{t}}\, dt$ **b.** $\int_0^{\pi/2} x\cos(x)\, dx$ **c.** $\int \sqrt{1-x^2}\, dx$

d. $\int_0^\infty e^{-y}\, dy$ **e.** $\int \frac{x^2+x+1}{x\left(x^2+1\right)}\, dx$ **f.** $\int_0^{\pi/4} \tan^2(z)\, dz$

3. Do any *three* (3) of **a–g**. *[15 = 3 × 5 each]*

a. Let $f(x) = x^2 + 1$ and compute $f'(1)$ using the limit definition of the derivative.

b. Find the arc-length of the curve $y = \frac{2}{3}x^{3/2}$, $0 \le x \le 3$.

c. Compute $\lim_{n\to\infty} \frac{n^2}{e^n}$.

d. Determine whether $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ converges absolutely or conditionally, or diverges.

e. Sketch the polar curve $r = \sec(\theta)$, $0 \le \theta \le \frac{\pi}{4}$, and find the area of the region between this curve and the origin.

f. Find the number b such that the average value of $y = 1 - x$ on $0 \le x \le b$ is $\frac{1}{2}$.

g. Find the interval of convergence of the power series $\sum_{n=0}^{\infty} 4^{-n}x^n$.

4. Consider the region below $y = \sqrt{1 - \frac{x^2}{4}}$ and above $y = 0$, for $-2 \le x \le 2$.

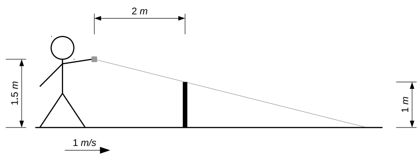
a. Sketch the region and find its area. *[6]*

b. Sketch the solid obtained by revolving this region about the x-axis and find its volume. *[6]*

Part M. Do any *two* (2) of **5–7**. *[28 = 2 × 14 each]*

5. Find the domain and any and all intercepts, vertical and horizontal asymptotes, and maximum, minimum, and inflection points of $f(x) = e^{-x^2}$, and sketch its graph.

6. Meredith, carrying a lamp 1.5 m above the ground, walks at 1 m/s along level ground directly toward a 1 m tall post at night. How is the length of the shadow cast by the post in the lamplight changing at the instant that the lamp is 2 m from the post?



7. Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n+1}-\sqrt{n-1}}{\sqrt{n\sqrt{n}}}$ converges or diverges.

Part N. Do *one* (1) of **8** or **9**. *[15 = 1 × 15]*

8. Let $f(x) = \sin\left(\dfrac{x}{2}\right)$.

a. Use Taylor's formula to find the Taylor series at 0 of $f(x)$. *[9]*

b. Find the radius and interval of convergence of this Taylor series. *[6]*

c. *[Bonus!]* Verify that the Taylor series at 0 of $f(x)$ actually converges to $f(x)$. *[1]*

9. Suppose $f(x)$ has $\displaystyle\sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}$ as its Taylor series at 0.

a. Find the radius and interval of convergence of this Taylor series. *[6]*

b. Use Taylor's formula to determine $f^{(n)}(0)$ for $n \geq 0$. *[9]*

c. *[Bonus!]* Find a formula, other than the series, for $f(x)$. *[1]*

[Total = 100]

Part O. Bonus problems! If you feel like it and have the time, do one or both of these.

○. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots = \frac{\pi^2}{6}$. Assuming this is so [which it is], what is the series $\displaystyle\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = 1 + \frac{1}{9} + \frac{1}{25} + \cdots$ equal to? *[1]*

⊙. Write a haiku touching on calculus or mathematics in general. *[1]*

What is a haiku?

seventeen in three:
five and seven and five of
syllables in lines

HAVE SOME FUN THIS SUMMER,
AND DROP BY NEXT YEAR TO TELL ME ABOUT IT!