Trent University

MATH 1101Y Test #1

16 October, 2012

Time: 50 minutes

Name: Solutions

Student Number: 3141592

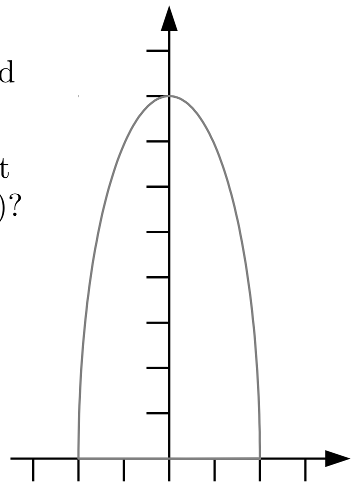
Question	Mark
1	____
2	____
3	____
Total	____ /30

Instructions

- *Show all your work.* Legibly, please!
- *If you have a question, ask it!*
- Use the back sides of the test sheets for rough work or extra space.
- You may use a calculator and an aid sheet.

1. Do any *two* (2) of **a**–**c**. *[10 = 2 × 5 each]*

a. Find the equation of the parabola with vertex at $(0, 8)$ and x-intercepts at $x = \pm 2$: that is, whose graph looks like:



b. Suppose you know that $f(x)$ is continuous at $x = a$. What can you conclude about the continuity of $g(x) = f(3x - 1)$?

c. Find all the horizontal asymptotes of $h(x) = \dfrac{1 - x^2}{1 + x^2}$.

SOLUTION TO **a**. The x-intercepts of the parabola are the roots of the corresponding quadratic expression, which must therefore be

$$y = a(x-2)\,(x-(-2)) = a(x-2)(x+2) = ax^2 - 4a$$

for some constant a. We can determine a by using the fact the vertex of the parabola is at $(0, 8)$: $8 = a0^2 - 4a = -4a$, so $a = 8/(-4) = -2$. Thus the equation of the parabola is $y = -2x^2 - 4(-2) = -2x^2 + 8$. ∎

SOLUTION TO **b**. Since $f(x)$ is continuous at $x = a$, $g(x) = f(3x-1)$, and $h(x) = 3x-1$ is continuous everywhere, it follows that $g(x)$ is continuous at the point x for which $3x - 1 = a$, that is, at $x = \dfrac{a+1}{3}$. (Recall that $g(x) = f\,(h(x))$ is continuous at x if $h(x)$ is continuous at x and $f(u)$ is continuous at $u = h(x)$.) ∎

SOLUTION TO **c**. We check the relevant limits:

$$\lim_{x\to+\infty} \frac{1-x^2}{1+x^2} = \lim_{x\to+\infty} \frac{1-x^2}{1+x^2} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}} = \lim_{x\to+\infty} \frac{\frac{1}{x^2} - 1}{\frac{1}{x^2} + 1} = \frac{0-1}{0+1} = -1 \quad \text{and}$$

$$\lim_{x\to-\infty} \frac{1-x^2}{1+x^2} = \lim_{x\to-\infty} \frac{1-x^2}{1+x^2} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}} = \lim_{x\to-\infty} \frac{\frac{1}{x^2} - 1}{\frac{1}{x^2} + 1} = \frac{0-1}{0+1} = -1\,,$$

since $\frac{1}{x^2} \to 0$ as $x \to +\infty$ and as $x \to -\infty$. It follows that $h(x)$ has a horizontal asymptote of $y = -1$ in both directions. ∎

2. Do any *two* (2) of **a**–**c**. *[12 = 2 × 6 each]*

a. Find all the discontinuities of $f(x) = \dfrac{x^3 + 3x^2 - x - 3}{x^2 - 1}$ and sketch its graph.

b. Compute $\displaystyle\lim_{x \to 0} x \sin\left(\frac{1}{x}\right)$.

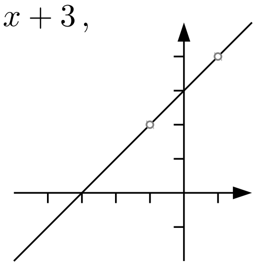
c. Use the limit definition of the derivative to find $f'(0)$ if $f(x) = (x+1)^2$.

SOLUTION TO **a**. $f(x) = \dfrac{x^3 + x^2 - x - 3}{x^2 - 1}$ is a rational function and is therefore continuous everywhere that its denominator is not equal to 0. $x^2 - 1 = 0$ when $x^2 = 1$, *i.e.* when $x = \pm 1$. Thus $f(x)$ is discontinuous at $x = -1$ and $x = 1$.

To graph $f(x)$, we first divide the denominator into the numerator in order simplify the expression we're dealing with. Since $x^3 + 3x^2 - x - 3 = (x+3)\left(x^2 - 1\right)$, it follows that

$$f(x) = \frac{x^3 + 3x^2 - x - 3}{x^2 - 1} = \frac{(x+3)\left(x^2 - 1\right)}{x^2 - 1} = x + 3\,,$$

at least when $x \neq \pm 1$. This means that the discontinuities at $x = \pm 1$ are removable (*i.e.* point) discontinuities and so the graph of $y = f(x)$ looks like:



Whew! ■

SOLUTION TO **b**. We'll use the Squeeze theorem to do this. Note that because $-1 \leq \sin\left(\frac{1}{x}\right) \leq 1$ for all $x \neq 0$, we have $-x = x \cdot (-1) \leq x \sin\left(\frac{1}{x}\right) = x \cdot 1 = x$ for all $x \neq 0$. Since both $\displaystyle\lim_{x \to 0}(-x) = 0$ and $\displaystyle\lim_{x \to 0} x = 0$, the Squeeze Theorem tells us that $\displaystyle\lim_{x \to 0} x \sin\left(\frac{1}{x}\right) = 0$. ■

NOTE: There is a small flaw in the reasoning in the solution to **b** above. What is it?

SOLUTION TO **c**. We plug $f(x) = (x+1)^2$ into the limit definition of $f'(0)$ and try hard:

$$\begin{aligned} f'(0) &= \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{((0+h)+1)^2 - (0+1)^2}{h} = \lim_{h \to 0} \frac{h^2 + 2h + 1^2 - 1^2}{h} \\ &= \lim_{h \to 0} \frac{h(h+2)}{h} = \lim_{h \to 0}(h+2) = 0 + 2 = 2 \qquad ■ \end{aligned}$$

3. Do *one* (1) of **a** or **b**. *[8]*

a. Find the inverse function of $f(x) = \dfrac{x}{1+x^2}$. What is the domain of $f^{-1}(x)$?

b. Verify that $\dfrac{2}{\tan(2x)} = \dfrac{1}{\tan(x)} - \tan(x)$.

SOLUTION TO **a**. As usual, we set $x = f(y)$ and try to solve for y:

$$x = f(y) = \frac{y}{1+y^2} \quad \Longrightarrow \quad x\left(1+y^2\right) = y \quad \Longrightarrow \quad xy^2 - y + x = 0$$

The last is a quadratic equation in y, some of whose coefficients involve x. Using the quadratic equation gives:

$$y = \frac{-(-1) \pm \sqrt{(-1)^2 - 4 \cdot x \cdot x}}{2x} = \frac{1 \pm \sqrt{1-4x^2}}{2x}$$

Thus $f^{-1}(x) = \dfrac{1 \pm \sqrt{1-4x^2}}{2x}$.

Observe that we actually get two possibilities for $f^{-1}(x)$, depending on whether we take the $+$ or the $-$ in the expression. Either way, the result makes sense so long as we are not dividing by zero and whatever is inside the square root is not negative. Thus the domain of $f^{-1}(x)$ consists of all x such that $x \neq 0$ and $1 - 4x^2 \geq 0$. Since

$$1 - 4x^2 \geq 0 \quad \Longrightarrow \quad 1 \geq 4x^2 \quad \Longrightarrow \quad \frac{1}{4} \geq x^2 \quad \Longrightarrow \quad |x| \leq \frac{1}{2} \quad \Longrightarrow \quad -\frac{1}{2} \leq x \leq -\frac{1}{2},$$

the domain of $f^{-1}(x)$ is $\left[-\frac{1}{2}, 0\right) \cup \left(0, \frac{1}{2}\right]$. Note that this does not include $x = 0$. ■

SOLUTION TO **b**. We'll use assorted trig identities to rewrite $\frac{2}{\tan(2x)}$ in terms of $\sin(x)$ and $\cos(x)$ and then see where we can go from there:

$$\begin{aligned}
\frac{2}{\tan(2x)} &= \frac{2}{\frac{\sin(2x)}{\cos(2x)}} = \frac{2\cos(2x)}{\sin(2x)} = \frac{2\left[\cos^2(x) - \sin^2(x)\right]}{2\sin(x)\cos(x)} \\
&= \frac{\cos^2(x)}{\sin(x)\cos(x)} - \frac{\sin^2(x)}{\sin(x)\cos(x)} = \frac{\cos(x)}{\sin(x)} - \frac{\sin(x)}{\cos(x)} \\
&= \frac{1}{\frac{\sin(x)}{\cos(x)}} - \tan(x) = \frac{1}{\tan(x)} - \tan(x) \qquad \blacksquare
\end{aligned}$$

NOTE: The trig happy can rewrite $\frac{1}{\tan(x)}$ as $\cot(x)$ if they so desire ...

[Total = 30]