

Mathematics 1101Y – Calculus I: Functions and calculus of one variable
TRENT UNIVERSITY, 2012–2013

Solutions to Assignment #2
Algebra and hyperbolic functions

Before attempting the questions below, please read through Section 1.6 of the text book for the definitions of the various hyperbolic functions. It might also be useful to skim though *Getting started with Maple 10* by Gilberto E. Urroz – read those parts concerned with solving equations more closely! – and perhaps keep it handy as a reference. You can find links to both documents on the MATH 1101Y web page. Maple's help facility may also come in handy, especially when trying to make out the intricacies of what the `solve` command and its options and variations do. Make use of the Maple labs, too, if possible.

1. Explain why every point on the parametric curve $x = \cosh(t)$ and $y = \sinh(t)$, $-\infty < t < \infty$, is on the hyperbola $x^2 - y^2 = 1$. What does t represent with respect to this hyperbola? [2]

HINT: To answer the question asked above, you might need to look up the hyperbolic functions somewhere other than the textbook.

SOLUTION. One could just look up the identities satisfied by the hyperbolic functions and observe that one of them is $\cosh^2(t) - \sinh^2(t) = 1$, but we'll do it the hard way below.

Recall that $\cosh(t) = \frac{e^t + e^{-t}}{2}$ and $\sinh(t) = \frac{e^t - e^{-t}}{2}$. It follows that if $x = \cosh(t)$ and $y = \sinh(t)$ for some t , then

$$\begin{aligned}x^2 - y^2 &= \cosh^2(t) - \sinh^2(t) = \left(\frac{e^t + e^{-t}}{2}\right)^2 - \left(\frac{e^t - e^{-t}}{2}\right)^2 \\&= \frac{1}{4} \left[(e^t)^2 + 2e^t e^{-t} + (e^{-t})^2 \right] - \frac{1}{4} \left[(e^t)^2 - 2e^t e^{-t} + (e^{-t})^2 \right] \\&= \frac{1}{4} (e^t)^2 + \frac{1}{4} \cdot 2e^t e^{-t} + \frac{1}{4} (e^{-t})^2 - \frac{1}{4} (e^t)^2 + \frac{1}{4} \cdot 2e^t e^{-t} - \frac{1}{4} (e^{-t})^2 \\&= \frac{1}{4} \cdot 4e^t e^{-t} = e^{t-t} = e^0 = 1.\end{aligned}$$

Thus every point (x, y) on the parametric curve $x = \cosh(t)$ and $y = \sinh(t)$ satisfies the equation $x^2 - y^2 = 1$, which means it is on the hyperbola. ■

We'll probably see the very occasional use of $\cosh(x)$, $\sinh(x)$, and/or $\tanh(x)$ in this course, so most of the rest of this assignment will, just to be different, be about $\operatorname{csch}(x) = \frac{1}{\sinh(x)} = \frac{2}{e^x - e^{-x}}$.

2. What are the domain and range of $\operatorname{csch}(x)$? Use Maple to plot the curve $y = \operatorname{csch}(x)$. [Please submit a printout of your worksheet.] On the basis of your plot, should (all of) $\operatorname{csch}(x)$ be invertible? [1]

SOLUTION. $\operatorname{csch}(x) = \frac{1}{\sinh(x)} = \frac{2}{e^x - e^{-x}}$ makes sense for all x that don't make the denominator 0. That is, it makes sense for all x except for any x such that

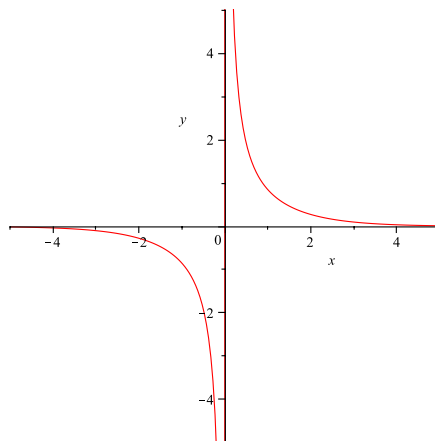
$$e^x - e^{-x} = 0 \iff e^x = e^{-x} \iff x = -x \iff x = 0.$$

Thus the domain of $\operatorname{csch}(x)$ is therefore $\{x \mid x \neq 0\} = (-\infty, 0) \cup (0, \infty)$.

As for the range, 0 is not in it: this would require that $\operatorname{csch}(x) = \frac{1}{\sinh(x)} = \frac{2}{e^x - e^{-x}} = 0$, which cannot happen unless $2 = 0 \dots$ However, every other real number is in the range: you can get arbitrarily large positive numbers out of $\operatorname{csch}(x)$ by plugging in positive x s close to 0, and get arbitrarily small ones by plugging in large positive x s. ($x = \operatorname{arcsch}(y)$, as obtained in the solution to **3** below, tells you exactly which x to plug in to get a given y . Note that the domain of $\operatorname{arcsch}(x)$ must be contained in the range of $\operatorname{csch}(x)$.)

Here's the plot:

> plot(2/(exp(x)-exp(-x)), x=-5..5, y=-5..5)



Since the entire graph appears to pass the horizontal line test, all of $\operatorname{csch}(x)$ should be invertible. ■

3. Find an expression – by hand! – for $\operatorname{arcsch}(x)$ in terms of the natural logarithm function. [2]

SOLUTION. As usual, we set $x = \operatorname{csch}(y) = \frac{2}{e^y - e^{-y}}$ and try to solve for y in terms of x . We'll first solve for e^y and then solve for y .

$$\begin{aligned} x = \frac{2}{e^y - e^{-y}} &\iff x(e^y - e^{-y}) = 2 &\iff xe^y - xe^{-y} = 2 \\ &\iff x(e^y)^2 - xe^{-y}e^y = 2e^y &\iff x(e^y)^2 - 2e^y - x = 0 \end{aligned}$$

We now use the quadratic formula to solve for e^y :

$$\begin{aligned} e^y &= \frac{-(-2) \pm \sqrt{(-2)^2 - 4 \cdot x \cdot (-x)}}{2 \cdot x} = \frac{2 \pm \sqrt{4 + 4x^2}}{2x} \\ &= \frac{2 \pm 2\sqrt{1 + x^2}}{2x} = \frac{1 \pm \sqrt{1 + x^2}}{x} \end{aligned}$$

It follows that

$$y = \ln(e^y) = \ln\left(\frac{1 \pm \sqrt{1 + x^2}}{x}\right).$$

(One could also write this as $y = \ln(1 \pm \sqrt{1 + x^2}) - \ln(x)$.) Note that this actually gives two solutions: $y = \ln\left(\frac{1 + \sqrt{1 + x^2}}{x}\right)$ and $y = \ln\left(\frac{1 - \sqrt{1 + x^2}}{x}\right)$, respectively. ■

4. Use **Maple** to find an expression for $\operatorname{arcsch}(x)$ in terms of the natural logarithm function. [Please submit a printout of your worksheet.] Is this the same as the one you obtained by hand? [2]

SOLUTION. Here we go:

> solve(x=2/(exp(y)-exp(-y)),y)

$$\ln\left(\frac{1 + \sqrt{1 + x^2}}{x}\right), \ln\left(-\frac{-1 + \sqrt{1 + x^2}}{x}\right)$$

It is not hard to see that the first solution **Maple** gives is identical to the first solution obtained in the solution to **3** above, and that the second given by **Maple** is a very minor rearrangement of the second obtained in the solution to **3**. ■

5. What are the domain and range of $\operatorname{arcsch}(x)$? [Your version of the expression, if it's different from **Maple**'s.] [2]

SOLUTION. Recall that $\ln(t)$ makes sense exactly when $t > 0$.

Looking at the first answer obtained in **3**, $\ln\left(\frac{1 + \sqrt{1 + x^2}}{x}\right)$, it is easy to see that the expression $\frac{1 + \sqrt{1 + x^2}}{x}$ makes sense and is positive exactly when $x > 0$.

Looking at the second answer obtained in **3**, $\ln\left(\frac{1 - \sqrt{1 + x^2}}{x}\right)$, note first that $1 - \sqrt{1 + x^2} < 0$ whenever $x \neq 0$. It follows that that the expression $\frac{1 - \sqrt{1 + x^2}}{x}$ makes sense and is positive exactly when $x < 0$.

Thus the domain of $\operatorname{arcsch}(x)$ is all $x \neq 0$, *i.e.* $(-\infty, 0) \cup (0, \infty)$.

As for the range, in both answers the input to \ln can be made arbitrarily large by taking x close enough to 0. Since $\ln(t) \rightarrow \infty$ as $t \rightarrow \infty$ (albeit very slowly), it follows that

the range of $\operatorname{arccsch}(x)$ extends to ∞ . On the other hand, in both answers the input to \ln can be made arbitrarily close to 0 by making x far enough away from 0. Since $\ln(t) \rightarrow -\infty$ as $t \rightarrow 0$, it follows that the range of $\operatorname{arccsch}(x)$ extends to $-\infty$. Note that in neither answer is it possible to find an x that makes the input to \ln be 1 (although you can get arbitrarily close), so 0 is not in the range $\operatorname{arccsch}(x)$.

Thus the range of $\operatorname{arccsch}(x)$ is also all $x \neq 0$, *i.e.* $(-\infty, 0) \cup (0, \infty)$. ■

6. Just for fun – and a mark too! – use Maple to find the roots of the general cubic equation, $ax^3 + bx^2 + cx + d = 0$. [Do not do this by hand, unless you have *way* too much time on your hands ...] [1]

SOLUTION. Here we go:

$$\begin{aligned}
 &> \text{solve}(a \cdot x^3 + b \cdot x^2 + c \cdot x + d = 0, x) \\
 &\frac{1}{6} \frac{1}{a} (36 c b a - 108 d a^2 - 8 b^3) \\
 &+ 12 \sqrt{3} \sqrt{4 c^3 a - c^2 b^2 - 18 c b a d + 27 d^2 a^2 + 4 d b^3 a}^{1/3} - \frac{2}{3} (3 c a \\
 &- b^2) / \left(a (36 c b a - 108 d a^2 - 8 b^3) \right. \\
 &+ 12 \sqrt{3} \sqrt{4 c^3 a - c^2 b^2 - 18 c b a d + 27 d^2 a^2 + 4 d b^3 a}^{1/3} \left. \right) - \frac{1}{3} \frac{b}{a}, \\
 &- \frac{1}{12} \frac{1}{a} (36 c b a - 108 d a^2 - 8 b^3) \\
 &+ 12 \sqrt{3} \sqrt{4 c^3 a - c^2 b^2 - 18 c b a d + 27 d^2 a^2 + 4 d b^3 a}^{1/3} + \frac{1}{3} (3 c a \\
 &- b^2) / \left(a (36 c b a - 108 d a^2 - 8 b^3) \right. \\
 &+ 12 \sqrt{3} \sqrt{4 c^3 a - c^2 b^2 - 18 c b a d + 27 d^2 a^2 + 4 d b^3 a}^{1/3} \left. \right) - \frac{1}{3} \frac{b}{a} \\
 &+ \frac{1}{2} 1 \sqrt{3} \left(\frac{1}{6} \frac{1}{a} (36 c b a - 108 d a^2 - 8 b^3) \right. \\
 &+ 12 \sqrt{3} \sqrt{4 c^3 a - c^2 b^2 - 18 c b a d + 27 d^2 a^2 + 4 d b^3 a}^{1/3} + \frac{2}{3} (3 c a \\
 &- b^2) / \left(a (36 c b a - 108 d a^2 - 8 b^3) \right. \\
 &+ 12 \sqrt{3} \sqrt{4 c^3 a - c^2 b^2 - 18 c b a d + 27 d^2 a^2 + 4 d b^3 a}^{1/3} \left. \right) \left. \right), \\
 &- \frac{1}{12} \frac{1}{a} (36 c b a - 108 d a^2 - 8 b^3) \\
 &+ 12 \sqrt{3} \sqrt{4 c^3 a - c^2 b^2 - 18 c b a d + 27 d^2 a^2 + 4 d b^3 a}^{1/3} + \frac{1}{3} (3 c a \\
 &- b^2) / \left(a (36 c b a - 108 d a^2 - 8 b^3) \right. \\
 &+ 12 \sqrt{3} \sqrt{4 c^3 a - c^2 b^2 - 18 c b a d + 27 d^2 a^2 + 4 d b^3 a}^{1/3} \left. \right) - \frac{1}{3} \frac{b}{a} \\
 &- \frac{1}{2} 1 \sqrt{3} \left(\frac{1}{6} \frac{1}{a} (36 c b a - 108 d a^2 - 8 b^3) \right. \\
 &+ 12 \sqrt{3} \sqrt{4 c^3 a - c^2 b^2 - 18 c b a d + 27 d^2 a^2 + 4 d b^3 a}^{1/3} + \frac{2}{3} (3 c a \\
 &- b^2) / \left(a (36 c b a - 108 d a^2 - 8 b^3) \right.
 \end{aligned}$$

■

REFERENCES

1. *Calculus: Early Transcendentals* (2nd Edition), by Jon Rogawski, W.H. Freeman, New York, 2012, ISBN-10: 1-4292-6009-2, ISBN-13: 978-1-4292-6009-1.
2. *Getting started with Maple 10*, by Gilberto E. Urroz (2005), which can found (pdf) at: <http://euclid.trentu.ca/math/sb/1101Y/2012-2013/GettingStartedMaple10.pdf>