Mathematics 1101Y – Calculus I: Functions and calculus of one variable
TRENT UNIVERSITY, 2011–2012

Solution to Assignment #5
The limit game

One of the things we've skipped over was the formal definition of limit, that is, how to pin down just what $\lim_{x \to a} f(x) = L$ really means. The usual definition is something like:

> DEFINITION. $\lim_{x \to a} f(x) = L$ exactly when for every $\varepsilon > 0$ there is a $\delta > 0$ such that for any x with $|x - a| < \delta$ we are guaranteed to have $|f(x) - L| < \varepsilon$ as well.

Informally, this means that no matter how close – that's the ε – you want $f(x)$ to get to L, you can make it happen by ensuring that x is close enough – that's the δ – to a. If this can always be done, $\lim_{x \to a} f(x) = L$; if not, then $\lim_{x \to a} f(x) \neq L$.

This definition works, but most people find it a little hard to understand and use at first. Here is less common definition equivalent to the one above that is cast in terms of a game:

> DEFINITION. The *limit game* for $f(x)$ at $x = a$ with target L is a three-move game played between two players A and B as follows:
>
> 1. A moves first, picking a small number $\varepsilon > 0$.
> 2. B moves second, picking another small number $\delta > 0$.
> 3. A moves third, picking an x that is within δ of a, *i.e.* $a - \delta < x < a + \delta$.
>
> To determine the winner, we evaluate $f(x)$. If it is within ε of the target L, *i.e.* $L - \varepsilon < f(x) < L + \varepsilon$, then player B wins; if not, then player A wins.
>
> With this idea in hand $\lim_{x \to a} f(x) = L$ means that player B has a winning strategy in the limit game for $f(x)$ at $x = a$ with target L; that is, if B plays it right, B will win no matter what A tries to do. (At least within the rules ... :-) Conversely, $\lim_{x \to a} f(x) \neq L$ means that player A is the one with a winning strategy in the limit game for $f(x)$ at $x = a$ with target L.

Your task in this assignment, should you choose to accept it, is to find such winning strategies:

1. Describe a winning strategy for B in the limit game for $f(x) = 2x + 1$ at $x = 1$ with target 3. Note that no matter what number ε A plays first, B must have a way to find a δ to play that will make it impossible for A to play an x that wins for A on the third move. *[5]*

SOLUTION. A winning strategy for B is to play $\delta = \frac{\varepsilon}{2}$ on move 2 if A plays $\varepsilon > 0$ on move 1. (Any $\delta > 0$ that's even smaller will also work.)

To see this, suppose that A plays some $\varepsilon > 0$ on move 1 and B responds by playing $\delta = \frac{\varepsilon}{2}$ on move 2. On move 3, A must select some x such that $1 - \delta < x < 1 + \delta$. No matter which such x A actually chooses, we get that

$$1 - \frac{\varepsilon}{2} = 1 - \delta < x < 1 + \delta = 1 + \frac{\varepsilon}{2} \quad \Longrightarrow \quad 2 - \varepsilon < 2x < 2 + \varepsilon$$
$$\Longrightarrow \quad 3 - \varepsilon < 2x + 1 < 3 + \varepsilon\,,$$

i.e. $L - \varepsilon < f(x) < L + \varepsilon$ for the function $f(x) = 2x + 1$ and the target $L = 3$. By the rules of the limit game, this means that B wins. ■

2. Describe a winning strategy for A in the limit game for $f(x) = 2x + 1$ at $x = 2$ with target 4. Note that A must pick an ε on the first move so that mo matter what δ B tries to play on the second move, A can still find an x to play on move three that wins for A. *[5]*

Solution. A winning strategy for A is to play $\varepsilon = \frac{1}{2}$ on move 1 and then, if B plays $\delta > 0$ on move 2, to play any x such that $2 - \min\left(\frac{1}{4}, \delta\right) < x < 2 + \min\left(\frac{1}{4}, \delta\right)$ on move 3.

To see this, suppose A plays $\varepsilon = \frac{1}{2}$ on move 1, B responds on move 2 with some $\delta > 0$, and A plays an x on move 3 such that $2 - \min\left(\frac{1}{4}, \delta\right) < x < 2 + \min\left(\frac{1}{4}, \delta\right)$. Then

$$2 - \delta \leq 2 - \min\left(\frac{1}{4}, \delta\right) < x < 2 + \min\left(\frac{1}{4}, \delta\right) \leq 2 - \delta\,,$$

so A's move is legal. It is also winning. Since we also have

$$2 - \frac{1}{4} \leq 2 - \min\left(\frac{1}{4}, \delta\right) < x < 2 + \min\left(\frac{1}{4}, \delta\right) \leq 2 - \frac{1}{4}\,,$$

it follows, multiplying by 2 throughout, that

$$3.5 = 4 - \frac{1}{2} < 2x < 4 + \frac{1}{2} = 4.5\,,$$

so, adding 1 throughout,

$$4.5 < 2x + 1 < 5.5\,.$$

Thus, since $L = 4$ and $\varepsilon = \frac{1}{2}$ here, we do *not* have

$$3.5 = 4 - \frac{1}{2} < 2x + 1 < 4 + \frac{1}{2} = 4.5\,,$$

so A wins by the rules of the limit game. ■