Mathematics 1101Y – Calculus I: Functions and calculus of one variable
TRENT UNIVERSITY, 2011–2012

Solution to Assignment #3
My name is Blond, Thames Blond.*

Thames Blond, playboy heir to the Pale River Ale fortune and not-so-secret agent of MI7, is cruising along one of Saskatchewan's famously straight roads in his BMW at its top speed of 200 *km/h*, approaching the point where one of Saskatchewan's famously straight railroads crosses the road at a right angle. The last car of a train is passing the crossing at a speed of 100 *km/h* just as Blond is 1 *km* away. At this instant, Blond spots the infamous Dr. Yes looking out the back of that last train car. He immediately swerves to follow†, keeping his BMW headed towards the last car of the train until he catches up.

1. If the train and Blond maintain their speeds of 100 *km/h* and 200 *km/h*, how far from the crossing does Blond catch up with the train? *[10]*

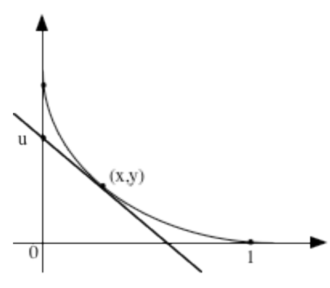
Hint: Supposing that the road lies along the x-axis and the railroad track along the y-axis, show that the BMW's path is the graph of a function satisfying the differential equation $2x\frac{d^2y}{dx^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$. Solve this equation (`Maple`'s `dsolve` command for solving differential equations may come in handy), assuming that $y = 0$ and $\frac{dy}{dx} = 0$ when $x = 1$, and take it from there . . .

Solution. The solution breaks down into two parts: getting to the differential equation mentioned in the hint and then solving it. *Since the first part proved more difficult than anticipated, its relative value was (originally 5/10) was vastly reduced (to 1/10) and full credit for the first part was given to any reasonable attempt to start a solution.* A complete solution of getting to the differential equation is nevertheless given below.

Just for fun – and to show off an interesting application of something we will learn later – we will obtain the differential equation with the help of an integral formula. This formula, which you can find in §8.1 of the textbook, states that the length of the curve given $y = f(x)$ for $a \le x \le b$ is

$$\text{arc-length} = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_a^b \sqrt{1 + (f'(x))^2}\, dx\,.$$

Setting up the axes as suggested in the hint, consider the situation at the instant that Blond's car is at the point (x, y):



* Apologies in advance to Ian Fleming and a certain prairie province.

† Remember that Saskatchewan is famously flat . . .

Note that the train is moving upward on the y-axis, Blond's BMW is pointed right at the end of the train, and his cars moves in the direction it is pointed. (We hope it does, anyway!). So the end of the train must be at the point $(0, u)$ where the tangent line to the curve traced out by the BMW intercepts the y-axis. The tangent line to the curve, at the point (x, y), has slope $\frac{dy}{dx}$. You can check for yourselves that the line with slope m passing through the point (c, d) intercepts the y-axis at the point $(0, d - cm)$. It follows that in our case

$$u = y - x\frac{dy}{dx}.$$

Note that at the instant Blond is at (x, y), the train has travelled a distance of u *km*, *i.e.* from $(0, 0)$ to $(0, u)$, since the instant that Blond spotted Dr. Yes and took off in pursuit. In the same period, Blond has travelled a distance equal to the arc length of the curve $y = y(x)$ traced by his BMW from x to 1. Plugging that into the appropriate integral formula gives

$$\text{arc length} = \int_x^1 \sqrt{1 - \left(\frac{dy}{dx}\right)^2}\, dx\,,$$

with a slight abuse of notation. (Namely?) Since Blond's BMW is moving at twice the speed of the train, it covers twice the distance in a given period of time, so $2u$ is equal to the arc length above. This gives us the differential/integral equation:

$$2\left(y - x\frac{dy}{dx}\right) = \int_x^1 \sqrt{1 - \left(\frac{dy}{dx}\right)^2}\, dx$$

Differentiating both sides of this equation, with the help of the Fundamental Theorem of Calculus on the right-hand side, gives

$$2\frac{dy}{dx} - 2 \cdot 1 \cdot \frac{dy}{dx} - 2x\frac{d^2y}{dx^2} = -\sqrt{1 - \left(\frac{dy}{dx}\right)^2}\,,$$

which, after simplifying, amounts to the desired differential equation:

$$2x\frac{d^2y}{dx^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

Whew!

Now all we need to do is to solve solve this differential equation. *Per* the set-up described in the hint, we get to assume that when $x = 1$, $y = 0$ and $\frac{dy}{dx} = 0$. Solving "second-order" differential equations (those involving second derivatives) is usually much harder than solving first-order differential equations. For us, this means that it will be harder to figure out just how to present it to `Maple` without choking on some point of trivia in how `Maple` works.

We can get around this problem, at least in part, by solving for $\frac{dy}{dx}$ – remember that $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$ – and then getting y by integrating $\frac{dy}{dx}$. If we set $z = \frac{dy}{dx}$, then $\frac{dz}{dx} = \frac{d^2y}{dx^2}$, so we can rewrite our differential equation as

$$2x\frac{dz}{dx} = \sqrt{1+z^2}\,.$$

This is a first-order differential equation with "initial condition" $z = \frac{dy}{dx} = 0$ when $x = 1$. We plug it into `Maple`'s specialized command `dsolve` for solving differential equations:

```
> dsolve(2*x*(diff(z(x),x))=sqrt(1+z(x)^2),z(1) = 0);
```

$$z(x) = \sinh\left(\frac{1}{2}\ln(x)\right)$$

That is, $\dfrac{dy}{dx} = z = \sinh\left(\dfrac{1}{2}\ln(x)\right)$.

We plug this (also first-order!) equation, with the initial condition $y = 0$ when $x = 1$ into `Maple` too:

```
> dsolve(diff(y(x),x)=sinh((1/2)*ln(x)),y(1)=0);
```

$$y(x) = \left(\frac{1}{3}(-3+x)\right)\sqrt{x} + \frac{2}{3}$$

We can now (finally!) answer the question. From the initial set-up, Blond catches up with the train when his car reaches the y-axis. When $x = 0$, $y = y(0) = \left(\frac{1}{3}(-3+0)\right)\sqrt{0} + \frac{2}{3} = \frac{2}{3}$. Thus Blond catches up with the train $\frac{2}{3}$ km from the crossing. ■

A SOLUTION BY HAND OF THE DIFFERENTIAL EQUATION.

$2x\frac{dz}{dx} = \sqrt{1+z^2}$ is a *separable* first-order differential equation, for which type there is a recipe to cook to order. Rearrange the equation to get

$$\frac{dz}{\sqrt{1+z^2}} = \frac{dx}{2x}\,,$$

(separating the variables on either side the = sign) and then integrate each side (with respect to z and x, respectively) to get

$$\ln\left(z+\sqrt{1+z^2}\right) = \int \frac{dz}{\sqrt{1+z^2}} = \int \frac{dx}{2x} = \ln\left(\sqrt{x}\right) + C\,,$$

where C is some constant. This can be done with a suitable trigonometric substitution on the left-hand side, or just by looking it up in the table of integrals on the inside cover of the textbook.

Exponentiating both sides of the resulting equation now gives

$$z + \sqrt{1+z^2} = e^{\ln\left(z+\sqrt{1+z^2}\right)} = e^{\ln\left(\sqrt{x}\right)+C} = K\sqrt{x}\,,$$

where $K = e^C$. Since $z = \frac{dy}{dx} = 0$ when $x = 1$, it follows that $K = 1$.

We now have to solve $z + \sqrt{1+z^2} = \sqrt{x}$ for z in terms of x. Squaring both sides gives

$$z^2 + 2z\sqrt{1+z^2} + \left(1+z^2\right) = x\,,$$

which unfortunately still involves $\sqrt{1+z^2}$. We get around this by rearranging the equation to get

$$2z\sqrt{1+z^2} = x - 2z^2 - 1\,,$$

and then squaring to get

$$4z^2\left(1+z^2\right) = 4z^2 + 4z^4 = x^2 - 4xz^2 - 2x + 4z^4 + 4z^2 + 1\,,$$

which simplifies to

$$0 = x^2 - 4xz^2 - 2x + 1\,.$$

Rearranging now gives

$$4xz^2 = x^2 - 2x + 1 = (x-1)^2\,,$$

from which it follows that

$$\frac{dy}{dx} = z = \frac{x-1}{2\sqrt{x}} = \frac{\sqrt{x}}{2} - \frac{1}{2\sqrt{x}}\,.$$

[Whew!]

We still have to integrate $z = \frac{dy}{dx}$ to get y as a function of x:

$$y = \int \left(\frac{\sqrt{x}}{2} - \frac{1}{2\sqrt{x}}\right)\,dx = \frac{x^{3/2}}{3} - \sqrt{x} + D$$

Since $y = 0$ when $x = 1$, a little arithmetic tells us that $D = \frac{2}{3}$, so

$$y = \frac{x^{3/2}}{3} - \sqrt{x} + \frac{2}{3}\,.$$

This, allowing for a little algebra, is what `Maple` gave us. □