Mathematics 1101Y – Calculus I: functions and calculus of one variable
TRENT UNIVERSITY, 2011–2012
Solutions to the Final Examination

Time: *09:00–12:00, on Friday, 14 April, 2011.* *Brought to you by* Стефан Біланюк.

Instructions: Do parts **U**, **V**, and **W**, and, if you wish, part **B**. Show all your work and justify all your answers. *If in doubt about something,* **ask!**

Aids: Calculator; one aid sheet (all sides!); one (1) brain (with $\leq 10^{10^{10}}$ neurons).

Part U. Do all four (4) of **1**–**4**.

1. Compute $\dfrac{dy}{dx}$ as best you can in any *three* (3) of **a**–**f**. *[15 = 3 × 5 each]*

a. $\ln(x+y)=1$ **b.** $y = x\arctan(x)$ **c.** $y=\sin\left(x^2\right)$

d. $\begin{array}{l} y=t^2-1 \\ x=2t+1 \end{array}$ **e.** $y=\displaystyle\int_0^x w^\pi\,dw$ **f.** $y=\dfrac{x^2-1}{x^2+1}$

SOLUTION TO **a**. $\ln(x+y)=1 \Longleftrightarrow x+y=0 \Longleftrightarrow y=-x$, so $\dfrac{dy}{dx}=\dfrac{d}{dx}(-x)=-1$. ■

SOLUTION TO **b**. Using the Product Rule,

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\left(x\arctan(x)\right) = \frac{dx}{dx}\cdot\arctan(x)+x\cdot\frac{d}{dx}\arctan(x) \\ &= 1\cdot\arctan(x)+x\cdot\frac{1}{1+x^2} = \arctan(x)+\frac{x}{1+x^2}\,. \qquad \blacksquare\end{aligned}$$

SOLUTION TO **c**. Using the Chain and Power Rules,

$$\frac{dy}{dx}=\frac{d}{dx}\sin\left(x^2\right)=\cos\left(x^2\right)\cdot\frac{d}{dx}x^2=\cos\left(x^2\right)\cdot 2x = 2x\cos\left(x^2\right)\,. \qquad \blacksquare$$

SOLUTION TO **d**. $\dfrac{dy}{dx}=\dfrac{\frac{dy}{dt}}{\frac{dx}{dt}}=\dfrac{\frac{d}{dt}\left(t^2-1\right)}{\frac{d}{dt}\left(2t+1\right)}=\dfrac{2t}{2}=t=\dfrac{x-1}{2}$. ■

SOLUTION TO **e**. Using the Fundamental Theorem of Calculus,

$$\frac{dy}{dx}=\frac{d}{dx}\int_0^x w^\pi\,dw = x^\pi\,. \qquad \blacksquare$$

SOLUTION TO **f**. Using the Quotient and Power Rules,

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\left(\frac{x^2-1}{x^2+1}\right)=\frac{\frac{d}{dx}\left(x^2-1\right)\cdot\left(x^2+1\right)-\left(x^2-1\right)\cdot\frac{d}{dx}\left(x^2+1\right)}{\left(x^2+1\right)^2} \\ &= \frac{2x\left(x^2+1\right)-\left(x^2-1\right)2x}{\left(x^2+1\right)^2}=\frac{2x^3+2x-2x^2+2x}{\left(x^2+1\right)^2}=\frac{4x}{\left(x^2+1\right)^2}\,. \qquad \blacksquare\end{aligned}$$

2. Evaluate any *three* (3) of the integrals **a**–**f**. *[15 = 3 × 5 each]*

$$\textbf{a.}\ \int \frac{1}{\sqrt{9-x^2}}\,dx \qquad \textbf{b.}\ \int_0^\infty e^{-x}\,dx \qquad \textbf{c.}\ \int \frac{e^s}{e^{2s}+1}\,ds$$

$$\textbf{d.}\ \int_1^e \ln(x)\,dx \qquad \textbf{e.}\ \int \frac{3x-3}{x^2+x-2}\,dx \qquad \textbf{f.}\ \int_0^{\pi/4} \tan(t)\,dt$$

Solution to **a**. We will use the trigonometric substitution $x = 3\sin(\theta)$, so $dx = 3\cos(\theta)\,d\theta$ and $\theta = \arcsin\left(\frac{x}{3}\right)$.

$$\begin{aligned}\int \frac{1}{\sqrt{9-x^2}}\,dx &= \int \frac{1}{\sqrt{9-9\sin^2(\theta)}} 3\cos(\theta)\,d\theta = \int \frac{3\cos(\theta)}{3\sqrt{1-\sin^2(\theta)}}\,d\theta \\ &= \int \frac{\cos(\theta)}{\sqrt{\cos^2(\theta)}}\,d\theta = \int \frac{\cos(\theta)}{\cos(\theta)}\,d\theta \\ &= \int 1\,d\theta = \theta + C = \arcsin\left(\frac{x}{3}\right) + C \qquad \blacksquare\end{aligned}$$

Solution to **b**. This is an improper integral, so we will need to take a limit.

$$\int_0^\infty e^{-x}\,dx = \lim_{t\to\infty} \int_0^t e^{-x}\,dx = \lim_{t\to\infty} \int_0^{-t} e^u(-1)\,du$$

(Using the substitution $u = -x$, so $dx = (-1)\,du$ and $\begin{smallmatrix} x & 0 & t \\ u & 0 & -t \end{smallmatrix}$.)

$$= \lim_{t\to\infty} -e^u\Big|_0^{-t} = \lim_{t\to\infty} \left(\left(-e^{-t}\right) - \left(-e^0\right)\right) = \lim_{t\to\infty} \left(-e^{-t}+1\right) = -0+1 = 1\,,$$

since $e^{-t} \to 0$ as $t \to \infty$. $\blacksquare$

Solution to **c**. We will use the substitution $u = e^s$, so $du = e^s\,ds$. Note that $e^{2s} = (e^s)^2$.

$$\int \frac{e^s}{e^{2s}+1}\,ds = \int \frac{1}{u^2+1}\,du = \arctan(u) + C = \arctan\left(e^s\right) + C \qquad \blacksquare$$

Solution to **d**. We'll use integration by parts, with $u = \ln(x)$ and $v' = 1$, so $u' = \dfrac{1}{x}$ and $v = x$.

$$\begin{aligned}\int_1^e \ln(x)\,dx &= \int_1^e uv'\,dx = uv|_1^e - \int_1^e u'v\,dx = x\ln(x)|_1^e - \int_1^e \frac{1}{x}x\,dx \\ &= e\ln(e) - 1\ln(1) - \int_1^e 1\,dx = e\cdot 1 - 1\cdot 0 - x|_1^e = e - 0 - (e-1) = 1 \qquad \blacksquare\end{aligned}$$

Solution to **e**. The key is to factor the denominator: $x^2+x-2 = (x-1)(x+2)$. (If necessary, one could find the roots of x^2+x-2 by using the quadratic formula.) Having factored it, one could use partial fractions, but in this case there is a shortcut:

$$\begin{aligned}\int \frac{3x-3}{x^2+x-2}\,dx &= \int \frac{3(x-1)}{(x-1)(x+2)}\,dx = \int \frac{3}{x+2}\,dx \\ &= 3\int \frac{1}{u}\,du = 3\ln(u) + C = 3\ln(x+2) + C\,,\end{aligned}$$

using the substitution $u = x + 2$, so $du = dx$. ■

SOLUTION TO **f**. Recall that $\tan(t) = \dfrac{\sin(t)}{\cos(t)}$. We will use the substitution $u = \cos(t)$, so $du = -\sin(t)\,dt$ and $\sin(t)\,dt = (-1)\,du$, and $\begin{array}{ccc} x & 0 & \pi/4 \\ u & 1 & 1/\sqrt{2} \end{array}$.

$$\int_0^{\pi/4} \tan(t)\,dt = \int_0^{\pi/4} \frac{\sin(t)}{\cos(t)}\,dt = \int_1^{1/\sqrt{2}} \frac{-1}{u}\,du = -\ln(u)\Big|_1^{1/\sqrt{2}}$$
$$= [-\ln(1)] - \left[-\ln\left(\frac{1}{\sqrt{2}}\right)\right] = -0 + \ln\left(2^{-1/2}\right) = -\frac{1}{2}\ln(2) \quad ■$$

3. Do any *three* (3) of **a**–**f**. *[15 = 3 × 5 each]*

a. Evaluate $\displaystyle\lim_{x\to 0} \frac{x}{\cos(x)}$ or show that the limit does not exist.

b. Find the area of the region enclosed by the polar curve $r = \sin(\theta)$ for $0 \le \theta \le \pi$. Sketch it, too!

c. Determine whether the series $\displaystyle\sum_{n=0}^{\infty} \frac{1}{2^n + n}$ converges or diverges.

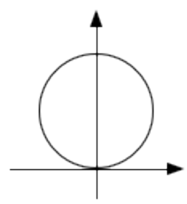
d. Use the Right-hand Rule to compute the definite integral $\displaystyle\int_2^3 x\,dx$.

e. Find the arc-length of the curve $y = \dfrac{2}{3}x^{3/2}$, $0 \le x \le 1$.

f. Find the interval of convergence of the power series $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n}$.

SOLUTION TO **a**. $\displaystyle\lim_{x\to 0} \frac{x}{\cos(x)} = \frac{0}{\cos(0)} = \frac{0}{1} = 0$. [They don't get much easier! :-)] ■

SOLUTION TO **b**. Here is a sketch of the region: (It turns out to be a circle of radius $\frac{1}{2}$ with its centre at the point $\left(0, \frac{1}{2}\right)$.)



The area of this region is given by

$$\int_0^{\pi} \frac{1}{2}r^2\,d\theta = \frac{1}{2}\int_0^{\pi} \sin^2(\theta)\,d\theta = \frac{1}{2}\int_0^{\pi} \frac{1}{2}\left(1 - \cos(2\theta)\right)\,d\theta = \frac{1}{4}\int_0^{2\pi} \left(1 - \cos(u)\right)\frac{1}{2}\,du$$
$$= \frac{1}{8}\left(u - \sin(u)\right)\Big|_0^{2\pi} = \frac{1}{8}(2\pi - 0) - \frac{1}{8}(0 - 0) = \frac{\pi}{4},$$

using the trigonometric identity $\cos(2\theta) = 1 - 2\sin^2(\theta)$ and the substitution $u = 2\theta$, so $du = 2d\theta$ and $d\theta = \frac{1}{2}du$, while $\begin{array}{ccc} \theta & 0 & \pi \\ u & 0 & 2\pi \end{array}$. ■

SOLUTION TO **c**. Since $0 < \dfrac{1}{2^n+n} \leq \dfrac{1}{2^n} = \left(\dfrac{1}{2}\right)^n$ for each $n \geq 0$, and $\displaystyle\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n$ converges (because it is a geometric series with common ratio $\frac{1}{2} < 1$), it follows that $\displaystyle\sum_{n=0}^{\infty}\frac{1}{2^n+n}$ converges by the (Basic) Comparison Test. ■

SOLUTION TO **d**. We will plug $a=2$, $b=3$, and $f(x)=x$ into the Right-Hand Rule formula $\displaystyle\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{b-a}{n}f\left(a+\frac{b-a}{n}i\right)$ and chug away:

$$\begin{aligned}
\int_2^3 x\,dx &= \lim_{n\to\infty}\sum_{i=1}^{n}\frac{3-2}{n}f\left(2+\frac{3-2}{n}i\right) = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{n}f\left(2+\frac{1}{n}i\right)\\
&= \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}f\left(2+\frac{i}{n}\right) = \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\left(2+\frac{i}{n}\right)\\
&= \lim_{n\to\infty}\frac{1}{n}\left(\left[\sum_{i=1}^{n}2\right]+\left[\sum_{i=1}^{n}\frac{i}{n}\right]\right) = \lim_{n\to\infty}\frac{1}{n}\left(2n+\frac{1}{n}\left[\sum_{i=1}^{n}i\right]\right)\\
&= \lim_{n\to\infty}\frac{1}{n}\left(2n+\frac{1}{n}\cdot\frac{n(n+1)}{2}\right) = \lim_{n\to\infty}\frac{1}{n}\left(2n+\frac{(n+1)}{2}\right)\\
&= \lim_{n\to\infty}\frac{1}{n}\left(\frac{5}{2}n+\frac{1}{2}\right) = \lim_{n\to\infty}\left(\frac{5}{2}+\frac{1}{2n}\right) = \frac{5}{2}+0 = \frac{5}{2},
\end{aligned}$$

since $\dfrac{1}{2n}\to 0$ as $n\to\infty$. ■

SOLUTION TO **e**. If $y = \dfrac{2}{3}x^{3/2}$, then $\dfrac{dy}{dx} = \dfrac{2}{3}\cdot\dfrac{3}{2}x^{1/2} = x^{1/2}$. Pluggin this into the arc-length formula along with $0 \leq x \leq 1$ gives:

$$\text{arc-length} = \int_0^1 ds = \int_0^1 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_0^1 \sqrt{1+\left(x^{1/2}\right)^2}\,dx = \int_0^1 \sqrt{1+x}\,dx$$

Substitute $u = x+1$, so $du = dx$ and $\begin{smallmatrix} x & 0 & 1 \\ u & 1 & 2 \end{smallmatrix}$

$$\begin{aligned}
&= \int_1^2 \sqrt{u}\,du = \int_1^2 u^{1/2}\,du = \left.\frac{u^{3/2}}{3/2}\right|_1^2 = \left.\frac{2}{3}u^{3/2}\right|_1^2 = \frac{2}{3}2^{3/2} - \frac{2}{3}1^{3/2}\\
&= \frac{2}{3}\left(2\sqrt{2}-1\right).
\end{aligned}$$

■

SOLUTION TO **f**. We will use the Ratio Test to find the radius of convergence.

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{\frac{x^{n+1}}{n+1}}{\frac{x^n}{n}}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{n+1}\cdot\frac{x^n}{n}\right| = \lim_{n\to\infty}\left|x\frac{n+1}{n}\right|\\
&= |x|\lim_{n\to\infty}\left(1+\frac{1}{n}\right) = |x|(1+0) = |x|
\end{aligned}$$

Thus, by the Ratio Test, the series converges if $|x| < 1$ and diverges if $|x| > 1$, so its radius of convergence is $R = 1$.

To determine the interval of convergence we check whether the series converges or diverges at the endpoints $x = \pm R = \pm 1$. Plugging in $x = 1$ gives $\sum_{n=0}^{\infty} \frac{1}{n}$, *i.e.* the harmonic series, which diverges by the p-Test since it has $p = 1 \leq 1$; plugging in $x = -1$ gives $\sum_{n=0}^{\infty} \frac{(-1)^n}{n}$, *i.e.* the alternating harmonic series, which converges:

i. Since $\frac{1}{n} > 0$ for all $n > 0$ and $(-1)^n$ alternates sign, $\frac{(-1)^n}{n}$ alternates sign.

ii. $\left|\frac{(-1)^{n+1}}{n}\right| = \frac{1}{n+1} < \frac{1}{n} = \left|\frac{(-1)^n}{n}\right|$ since $n + 1 > n$ for all n.

iii. $\lim_{n\to\infty} \left|\frac{(-1)^n}{n}\right| = \lim_{n\to\infty} \frac{1}{n} = 0$.

It follows by the Alternating Series Test that $\sum_{n=0}^{\infty} \frac{(-1)^n}{n}$ converges. Thus the interval of convergence of $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n}$ is $[-1, 1)$.

The sharp-eyed will have noticed the glitch at $n = 0$ in the definition of the given power series, which has been ignored throughout this solution ... ■

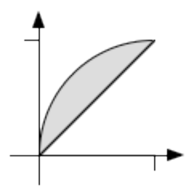
4. Consider the region between $y = \sqrt{x}$ and $y = x$ for $0 \leq x \leq 1$.

a. Sketch this region. *[1]*

b. Sketch the solid obtained by revolving this region about the y-axis. *[1]*

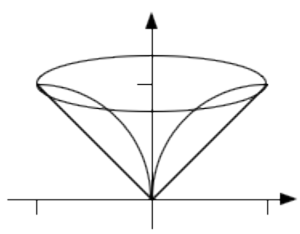
c. Find the volume of this solid. *[8]*

Solution to **a**. Here is sketch of the region:



■

Solution to **b**. Here is sketch of the solid:



■

Solution to **c**. *i.* *(Using washers.)* Since the region was revolved about the y-axis, the washers are stacked vertically, so we will have to integrate with respect to y. Note that $0 \leq y \leq 1$ over the given region. The outside radius of the washer at y comes from the line $y = x$ and is given by $R = x - 0 = x = y$, while the inside radius of the washer at y

comes from the curve $y = \sqrt{x}$ (so $x = y^2$) and is given by $r = x - 0 = x = y^2$. It follows that the volume of the solid is given by:

$$\int_0^1 \pi\left(R^2 - r^2\right)\,dy = \int_0^1 \pi\left(y^2 - \left(y^2\right)^2\right)\,dy = \pi\int_0^1 \left(R^2 - r^2\right)\,dy = \pi\int_0^1 \left(y^2 - y^4\right)\,dy$$
$$= \pi\left(\frac{y^3}{3} - \frac{y^5}{5}\right)\Big|_0^1 = \pi\left(\frac{1}{3} - \frac{1}{5}\right) - \pi\left(\frac{0}{3} - \frac{0}{5}\right) = \frac{2}{15}\pi \qquad \Box$$

ii. (Using cylindrical shells.) Since the region was revolved about the y-axis, we must integrate with respect to x to move perpendicular to the shells. Note that $0 \leq x \leq 1$ over the given region. The radius of the shell at x is just $r = x - 0 = x$, while its height is given by $h = \sqrt{x} - x = x^{1/2} - x$. It follows that the volume of the solid is given by:

$$\int_0^1 2\pi rh\,dx = 2\pi\int_0^1 x\left(x^{1/2} - x\right)\,dx = 2\pi\int_0^1 \left(x^{3/2} - x^2\right)\,dx = 2\pi\left(\frac{2x^{5/2}}{5} - \frac{x^3}{3}\right)\Big|_0^1$$
$$= 2\pi\left(\frac{2}{5} - \frac{1}{3}\right) - 2\pi\left(\frac{0}{5} - \frac{0}{3}\right) = 2\pi\frac{1}{15} = \frac{2}{15}\pi \qquad \blacksquare$$

Part V. Do any *two* (2) of **5**–**7**. *[30 = 2 × 15 each]*

5. The area of a rectangle with sides of lengths a and b is ab. Suppose the area of the rectangle is growing at a rate of 10 cm^2/s and a is increasing at a rate of 1 cm/s.

a. How is b changing at the instant that $a = 10$ cm and $b = 20$ cm? *[10]*

b. Suppose b is increasing at a rate of 1 cm/s at some instant. Find possible values for a and b at this instant or show that there are no such values. *[5]*

Solution to **a**. Let $A = ab$ denote the area of the rectangle. We are given that $\frac{dA}{dt} = 10$ cm^2/s and that $\frac{da}{dt} = 1$ cm/s. On the other hand, since $A = ab$, $\frac{dA}{dt} = \frac{d}{dt}(ab) = \frac{da}{dt} \cdot b + a \cdot \frac{db}{dt}$. At the instant that $a = 10$ cm and $b = 20$ cm we therefore have

$$10 = \frac{dA}{dt} = \frac{da}{dt} \cdot b + a \cdot \frac{db}{dt} = 1 \cdot 20 + 10 \cdot \frac{db}{dt},$$

so $\frac{db}{dt} = \frac{10 - 20}{10} = -1$ cm/s at this instant, *i.e.* b is decreasing at a rate of 1 cm/s. $\blacksquare$

Solution to **b**. Suppose b is increasing at a rate of 1 cm/s at some instant, *i.e.* $\frac{db}{dt} = 1$ cm/s. Plugging this into $\frac{dA}{dt} = \frac{d}{dt}(ab) = \frac{da}{dt} \cdot b + a \cdot \frac{db}{dt}$ along with $\frac{dA}{dt} = 10$ cm^2/s and $\frac{da}{dt} = 1$ cm/s gives us:

$$10 = \frac{dA}{dt} = \frac{da}{dt} \cdot b + a \cdot \frac{db}{dt} 1 \cdot b + a \cdot 1 = a + b$$

Thus any combination of a and b adding up to 10 is, in principle, possible at this instant. For example, $a = b = 5$ cm would do the job. ■

6. Find all the intercepts, maximum, minimum, and inflection points, and all the vertical and horizontal asymptotes of $f(x) = \dfrac{x^2-1}{x^2+1}$, and sketch its graph.

SOLUTION. We run through the usual checklist:

i. (Domain) Since $f(x)$ is a rational function whose denominator, $x^2+1 \geq 1$, is never 0, it is defined (and continous and differentiable) for all $x \in \mathbb{R}$.

ii. (Intercepts) $f(0) = \dfrac{0^2-1}{0^2+1} = -1$, so the function has a y-intercept of 0. Note that $f(x) = \dfrac{x^2-1}{x^2+1} = 0$ exactly when $x^2-1=0$, namely $x = \pm 1$. Thus $f(x)$ has x-intercepts at -1 and 1.

iii. (Vertical asymptotes) Since $f(x)$, as noted in *i* above, is defined and continuous for all x, it has no vertical asymptotes.

iv. (Horizontal asymptotes) Since

$$\lim_{x\to+\infty} \frac{x^2-1}{x^2+1} = \lim_{x\to+\infty} \frac{x^2-1}{x^2+1} \cdot \frac{1/x^2}{1/x^2} = \lim_{x\to+\infty} \frac{1-\frac{1}{x^2}}{1+\frac{1}{x^2}} = \frac{1-0}{1+0} = 1 \quad \text{and}$$
$$\lim_{x\to-\infty} \frac{x^2-1}{x^2+1} = \lim_{x\to-\infty} \frac{x^2-1}{x^2+1} \cdot \frac{1/x^2}{1/x^2} = \lim_{x\to-\infty} \frac{1-\frac{1}{x^2}}{1+\frac{1}{x^2}} = \frac{1-0}{1+0} = 1\,,$$

$y = f(x)$ has a horizontal asymptote of $y = 1$ in both directions. Note that $\dfrac{1}{x^2} \to 0$ as $x \to \pm\infty$.

v. (Critical points and maxima and minima) Note that

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left(\frac{x^2-1}{x^2+1}\right) = \frac{\frac{d}{dx}\left(x^2-1\right)\cdot\left(x^2+1\right) - \left(x^2-1\right)\cdot\frac{d}{dx}\left(x^2+1\right)}{\left(x^2+1\right)^2} \\ &= \frac{2x\left(x^2+1\right)-\left(x^2-1\right)2x}{\left(x^2+1\right)^2} = \frac{2x^3+2x-2x^3+2x}{\left(x^2+1\right)^2} = \frac{4x}{\left(x^2+1\right)^2}\,. \end{aligned}$$

It follows that $f'(x) = 0 \iff 4x = 0 \iff x = 0$. Since $f(x)$ and $f'(x)$ are defined and continuous for all x, this is the only critical point. Building the usual table,

x	$(-\infty, 0)$	0	$(0, \infty)$
$f'(x)$	$-$	0	$+$
$f(x)$	$\downarrow$	min	$\uparrow$

,

tells us that this critical point is a (local and absolute) minimum.

vi. (Curvature and inflection points) Note that

$$
\begin{aligned}
f''(x) = \frac{d}{dx}\left(\frac{4x}{(x^2+1)^2}\right) &= \frac{\frac{d}{dx}(4x)\cdot\left(x^2+1\right)^2 - 4x\cdot\frac{d}{dx}\left(x^2+1\right)^2}{\left((x^2+1)^2\right)^2} \\
&= \frac{4\left(x^2+1\right)^2 - 4x\cdot 2\left(x^2+1\right)2x}{(x^2+1)^4} = \frac{4\left(x^2+1\right) - 4x\cdot 2\cdot 2x}{(x^2+1)^3} \\
&= \frac{4x^2+4-16x^2}{(x^2+1)^3} = \frac{4\left(1-3x^2\right)}{(x^2+1)^3}\,.
\end{aligned}
$$

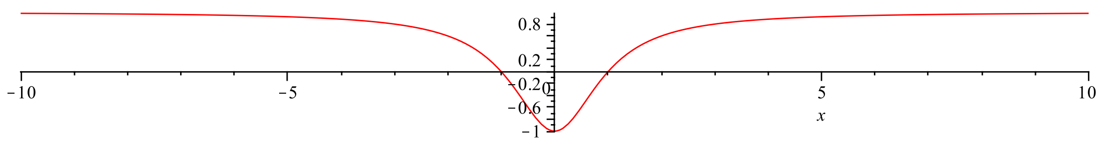
It follows that $f''(x) = 0 \iff 1 - 3x^2 = 0 \iff x = \pm\dfrac{1}{\sqrt{3}}$. Since $f(x)$, $f'(x)$, and $f''(x)$ are defined and continuous for all x, there are no other candidates for inflection points. Building the usual table,

x	$\left(-\infty, -\frac{1}{\sqrt{3}}\right)$	$-\frac{1}{\sqrt{3}}$	$\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$	$\frac{1}{\sqrt{3}}$	$\left(\frac{1}{\sqrt{3}}, \infty\right)$
$f''(x)$	$-$	0	$+$	0	$-$
$f(x)$	$\frown$	infl. pt.	$\smile$	infl. pt.	$\frown$

,

tells us that both or our candidates are, in fact, inflection points.

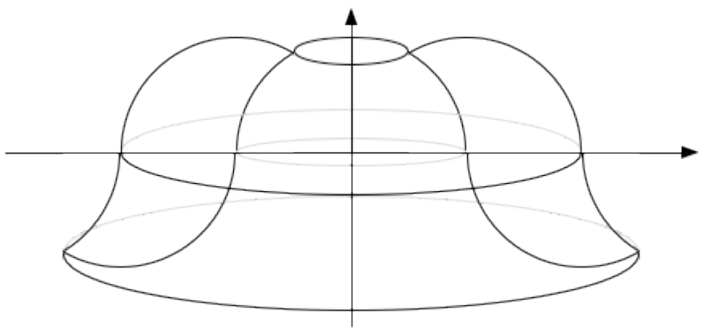
vii. (The graph) Here it is, with a little help from `Maple`:



That's all folks! (Whew! :-) ■

7. Sketch the solid obtained by revolving the region between $y = \cos(x)$ and $y = \sin(x)$, for $\frac{\pi}{4} \le x \le \frac{5\pi}{4}$, about the y-axis and find its volume using cylindrical shells.

SOLUTION. Here is a crude sketch of the solid.



Since we revolved the given region about the y-axis and are using cylindrical shells, we need to integrate with respect to x. The cylindrical shell at x will have radius $r = x - 0 = x$ and height $h = \sin(x) - \cos(x)$. (Notice that for $\frac{\pi}{4} \leq x \leq \frac{5\pi}{4}$, we have $\sin(x) \geq \cos(x)$ and that $\sin(x) = \cos(x)$ at the endpoints.) It follows that its volume is:

$$V = \int_{\pi/4}^{5\pi/4} 2\pi r h \, dx = 2\pi \int_{\pi/4}^{5\pi/4} x\left(\sin(x) - \cos(x)\right) \, dx$$

Use parts with $u = x$ and $v' = \sin(x) - \cos(x)$, so $u' = 1$ and $v = -\cos(x) - \sin(x)$.

$$\begin{aligned}
&= 2\pi x\left(-\cos(x) - \sin(x)\right)\Big|_{\pi/4}^{5\pi/4} - 2\pi \int_{\pi/4}^{5\pi/4} 1\left(-\cos(x) - \sin(x)\right) \, dx \\
&= 2\pi \frac{5\pi}{4}(-1)\left(\cos\left(\frac{5\pi}{4}\right) + \sin\left(\frac{5\pi}{4}\right)\right) - 2\pi\frac{\pi}{4}(-1)\left(\cos\left(\frac{\pi}{4}\right) + \sin\left(\frac{\pi}{4}\right)\right) \\
&\qquad + 2\pi \int_{\pi/4}^{5\pi/4} \left(\cos(x) + \sin(x)\right) \, dx \\
&= -\frac{5\pi^2}{2}\left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right) + \frac{\pi^2}{2}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) + 2\pi\left(\sin(x) - \cos(x)\right)\Big|_{\pi/4}^{5\pi/4} \\
&= \frac{6\pi^2}{2} \cdot \frac{2}{\sqrt{2}} + 2\pi\left(\cos\left(\frac{5\pi}{4}\right) - \sin\left(\frac{5\pi}{4}\right)\right) - 2\pi\left(\cos\left(\frac{\pi}{4}\right) - \sin\left(\frac{\pi}{4}\right)\right) \\
&= \frac{6\pi^2}{\sqrt{2}} + 2\pi 0 - 2\pi 0 = 3\sqrt{2}\pi^2 \qquad \blacksquare
\end{aligned}$$

Part W. Do *one* (1) of **8** or **9**. *[15 = 1 × 15 each]*

8. Consider the power series $\displaystyle\sum_{n=1}^{\infty} \frac{3^{n+2}x^n}{n!}$.

a. Find the radius and interval of convergence of this power series. *[10]*

b. What function has this power series as its Taylor series at 0? *[5]*

Solution to **a**. We will use the Ratio Test to find the radius of convergence.

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{\frac{3^{(n+1)+2}x^{n+1}}{(n+1)!}}{\frac{3^{n+2}x^n}{n!}}\right| = \lim_{n\to\infty}\left|\frac{3^{n+3}x^{n+1}}{(n+1)!} \cdot \frac{n!}{3^{n+2}x^n}\right| \\
&= \lim_{n\to\infty}\left|\frac{3x}{n+1}\right| = |3x| \lim_{n\to\infty}\frac{1}{n+1} = |3x| \cdot 0 = 0
\end{aligned}$$

By the Ratio Test, since $0 < 1$, the series converges for all x, *i.e.* the radius of convergence is $R = \infty$. It follows that the interval of convergence is $(-\infty, \infty)$. $\blacksquare$

Solution to **b**. Recall that $e^t = \displaystyle\sum_{n=0}^{\infty}\frac{t^n}{n!}$. Here goes:

$$\sum_{n=1}^{\infty}\frac{3^{n+2}x^n}{n!} = 3^2\sum_{n=1}^{\infty}\frac{3^n x^n}{n!} = 9\sum_{n=1}^{\infty}\frac{(3x)^n}{n!} = 9\left(\sum_{n=0}^{\infty}\frac{(3x)^n}{n!} - 1\right) = 9\left(e^{3x} - 1\right) \qquad \blacksquare$$

9. Let $f(x) = \dfrac{1}{2x-1}$.

a. Use Taylor's formula to find the Taylor series at 0 of $f(x)$. *[8]*

b. Find the Taylor series at 0 of $f(x)$ without using Taylor's formula. *[2]*

c. Find the radius and interval of convergence of this Taylor series. *[5]*

Solution to **a**. To use Taylor's formula here we need to know $f^{(n)}(0)$ for all $n \geq 0$.

$$\begin{array}{ccc}
n & f^{(n)}(x) & f^{(n)}(0) \\
0 & (2x-1)^{-1} & -1 \\
1 & (-1)(2x-1)^{-2}2 & -1 \cdot 2 \\
2 & (-1)(-2)(2x-1)^{-3}2^2 & -1 \cdot 2 \cdot 2^2 \\
3 & (-1)(-2)(-3)2^3(2x-1)^{-4} & -1 \cdot 6 \cdot 2^3 \\
\vdots & \vdots & \vdots \\
n & (-1)^n n!(2x-1)^{-n-1}2^n & -1 \cdot n! \cdot 2^n \\
\vdots & \vdots & \vdots
\end{array}$$

Plugging this into Taylor's formula gives:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{-1 \cdot n! \cdot 2^n}{n!} x^n = -\sum_{n=0}^{\infty} 2^n x^n \qquad \blacksquare$$

Solution to **b**. We will use the formula for the sum of a geometric series, $a + ar + ar^2 + ar^3 + \cdots = \frac{a}{1-r}$, in reverse:

$$\begin{aligned}
\frac{1}{2x-1} = \frac{-1}{1-2x} &= -1 - 2x - (2x)^2 - (2x)^3 - \cdots \\
&= -(1 + 2x + 4x^2 + 8x^3 + \cdots) = -\sum_{n=0}^{\infty} 2^n x^n
\end{aligned}$$

Note that $a = -1$ and $r = 2x$ here. Since a function can be equal to at most one power series at 0, this must must be the Taylor series at 0 of $f(x)$. $\blacksquare$

Solution to **c**. From the answer to **b**, we know that the Taylor series at 0 of $f(x)$ is a geometric series with $r = 2x$. It follows that it converges when $|r| = |2x| < 1$, *i.e.* when $-\frac{1}{2} < x < \frac{1}{2}$, and diverges otherwise. Thus the interval of convergence is $\left(-\frac{1}{2}, \frac{1}{2}\right)$. $\blacksquare$

[Total = 100]

Part B. ~~Bear~~ Bonus problems! Do them (or not), if you feel like it.

2b. Suppose a number of circles are drawn on a piece of paper, dividing it up into regions whose borders are made up of circular arcs. Show that one can always colour these regions using only black and white so that no two regions that have a border arc in common have the same colour. *[2]*

Solution. For each point, count the number of circles it's in (or on). If this number is even, colour the point white; if it is odd, colour it black. Now explain why this does the job... :-) ■

$\neg$**2b.** Write a haiku touching on calculus or mathematics in general. *[2]*

haiku?

seventeen in three:
five and seven and five of
syllables in lines

I hope that you enjoyed the course. (Really! :-)
Have a fun summer!