# Mathematics 1101Y – Calculus I: Functions and calculus of one variable

Trent University, 2011–2012

## Assignment #5
## The limit game
*Due on Thursday, 8 March, 2012.*

One of the things we've skipped over was the formal definition of limit, that is, how to pin down just what $\lim_{x \to a} f(x) = L$ really means. The usual definition is something like:

DEFINITION. $\lim_{x \to a} f(x) = L$ exactly when for every $\varepsilon > 0$ there is a $\delta > 0$ such that for any $x$ with $|x - a| < \delta$ we are guaranteed to have $|f(x) - L| < \varepsilon$ as well.

Informally, this means that no matter how close – that's the $\varepsilon$ – you want $f(x)$ to get to $L$, you can make it happen by ensuring that $x$ is close enough – that's the $\delta$ – to $a$. If this can always be done, $\lim_{x \to a} f(x) = L$; if not, then $\lim_{x \to a} f(x) \neq L$.

This definition works, but most people find it a little hard to understand and use at first. Here is less common definition equivalent to the one above that is cast in terms of a game:

DEFINITION. The *limit game* for $f(x)$ at $x = a$ with target $L$ is a three-move game played between two players $A$ and $B$ as follows:

1. $A$ moves first, picking a small number $\varepsilon > 0$.
2. $B$ moves second, picking another small number $\delta > 0$.
3. $A$ moves third, picking an $x$ that is within $\delta$ of $a$, *i.e.* $a - \delta < x < a + \delta$.

To determine the winner, we evaluate $f(x)$. If it is within $\varepsilon$ of the target $L$, *i.e.* $L - \varepsilon < f(x) < L + \varepsilon$, then player $B$ wins; if not, then player $A$ wins.

With this idea in hand $\lim_{x \to a} f(x) = L$ means that player $B$ has a winning strategy in the limit game for $f(x)$ at $x = a$ with target $L$; that is, if $B$ plays it right, $B$ will win no matter what $A$ tries to do. (At least within the rules ... :-) Conversely, $\lim_{x \to a} f(x) \neq L$ means that player $A$ is the one with a winning strategy in the limit game for $f(x)$ at $x = a$ with target $L$.

Your task in this assignment, should you choose to accept it, is to find such winning strategies:

1. Describe a winning strategy for $B$ in the limit game for $f(x) = 2x + 1$ at $x = 1$ with target 3. Note that no matter what number $\varepsilon$ $A$ plays first, $B$ must have a way to find a $\delta$ to play that will make it impossible for $A$ to play an $x$ that wins for $A$ on the third move. *[5]*

2. Describe a winning strategy for $A$ in the limit game for $f(x) = 2x + 1$ at $x = 2$ with target 4. Note that $A$ must pick an $\varepsilon$ on the first move so that mo matter what $\delta$ $B$ tries to play on the second move, $A$ can still find an $x$ to play on move three that wins for $A$. *[5]*