Mathematics 1101Y – Calculus I: functions and calculus of one variable
TRENT UNIVERSITY, 2010–2011
Solutions to the Final Examination

Part X. Do all three (3) of **1**–**3**.

1. Compute $\dfrac{dy}{dx}$ as best you can in any *three* (3) of **a**–**f**. *[15 = 3 × 5 each]*

a. $y = \cos\left(e^x\right)$ **b.** $y = \displaystyle\int_1^x e^t \ln(t)\, dt$ **c.** $y = x\ln(x)$

d. $y = \dfrac{\ln(x)}{x}$ **e.** $\arctan(x+y) = 0$ **f.** $\begin{matrix} x = e^t \\ y = e^{2t} \end{matrix}$

SOLUTIONS TO **1**.

a. Using the Chain Rule: $\dfrac{dy}{dx} = -\sin\left(e^x\right) \cdot \dfrac{d}{dx} e^x = -e^x \sin\left(e^x\right)$ $\square$

b. Using the Fundamental Theorem of Calculus: $\dfrac{dy}{dx} = e^x \ln(x)$ $\square$

c. Using the Product Rule: $\dfrac{dy}{dx} = \dfrac{dx}{dx} \cdot \ln(x) + x \cdot \dfrac{d}{dx}\ln(x) = \ln(x) + x \cdot \dfrac{1}{x} = \ln(x) + 1$ $\square$

d. Using the Quotient Rule:

$$\frac{dy}{dx} = \frac{\left(\frac{d}{dx}\ln(x)\right) \cdot x - \ln(x) \cdot \frac{dx}{dx}}{x^2} = \frac{\frac{1}{x} \cdot x - \ln(x) \cdot 1}{x^2} = \frac{1 - \ln(x)}{x^2} \quad \square$$

e. $\arctan(x+y) = 0 \Leftrightarrow x + y = 0 \Leftrightarrow y = -x \Rightarrow \dfrac{dy}{dx} = -1$ $\square$

Note: This can also be done using implicit differentiation.

f. $\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}} = \dfrac{\frac{d}{dt}e^{2t}}{\frac{d}{dt}e^t} = \dfrac{e^{2t} \cdot \frac{d}{dt}(2t)}{e^t} = \dfrac{2e^{2t}}{e^t} = \dfrac{2\left(e^t\right)^2}{e^t} = 2e^t = 2x$ $\square$

Note: This can also be done by observing that $y = e^{2t} = \left(e^t\right)^2 = x^2$ to begin with.

2. Evaluate any *three* (3) of the integrals **a**–**f**. *[15 = 3 × 5 each]*

a. $\displaystyle\int \frac{\ln(x)}{x}\, dx$ **b.** $\displaystyle\int \frac{1}{\sqrt{z^2 - 1}}\, dz$ **c.** $\displaystyle\int_1^4 \sqrt{x}\, dx$

d. $\displaystyle\int_0^1 \frac{x^2+2}{x^2+1}\, dx$ **e.** $\displaystyle\int \tan^2(w)\, dw$ **f.** $\displaystyle\int_0^{\ln(2)} e^{2t}\, dt$

SOLUTIONS TO **2**.

a. Use the substitution $u = \ln(x)$, so $du = \frac{1}{x}\, dx$:

$$\int \frac{\ln(x)}{x}\, dx = \int u\, du = u^2 + C = \left(\ln(x)\right)^2 + C \quad \square$$

b. Use the trigonometric substition $z = \sec(\theta)$, so $dz = \sec(\theta)\tan(\theta)\,d\theta$:

$$\begin{aligned}
\int \frac{1}{\sqrt{z^2-1}}\,dz &= \int \frac{1}{\sqrt{\sec^2(\theta)-1}}\,\sec(\theta)\tan(\theta)\,d\theta = \int \frac{1}{\sqrt{\tan^2(\theta)}}\,\sec(\theta)\tan(\theta)\,d\theta \\
&= \int \frac{1}{\tan(\theta)}\,\sec(\theta)\tan(\theta)\,d\theta = \int \sec(\theta)\,d\theta \\
&= \ln\left(\sec(\theta)+\tan(\theta)\right) + C = \ln\left(z+\sqrt{z^2-1}\right) + C \quad \square
\end{aligned}$$

c. Use the Power Rule for integration:

$$\begin{aligned}
\int_1^4 \sqrt{x}\,dx &= \int_1^4 x^{1/2}\,dx = \left.\frac{x^{3/2}}{3/2}\right|_1^4 = \left.\frac{2}{3}\left(\sqrt{x}\right)^3\right|_1^4 \\
&= \frac{2}{3}\left(\sqrt{4}\right)^3 - \frac{2}{3}\left(\sqrt{1}\right)^3 = \frac{2}{3}2^3 - \frac{2}{3} = \frac{14}{3} \quad \square
\end{aligned}$$

d. Observe that the integrand is a rational function in which the degree of the numerator is not less than the degree of the denominator. This means that we need to divide the denominator into the numerator first, which is very easy in this case:

$$\frac{x^2+2}{x^2+1} = \frac{\left(x^2+1\right)+1}{x^2+1} = \frac{x^2+1}{x^2+1} + \frac{1}{x^2+1} = 1 + \frac{1}{x^2+1}$$

Hence

$$\begin{aligned}
\int_0^1 \frac{x^2+2}{x^2+1}\,dx &= \int_0^1 \left(1+\frac{1}{x^2+1}\right) = \int_0^1 1\,dx + \int_0^1 \frac{1}{x^2+1}\,dx \\
&= x|_0^1 + \arctan(x)|_0^1 = [1-0] + [\arctan(1)-\arctan(0)] \\
&= 1 + \left[\frac{\pi}{4} - 0\right] = 1 + \frac{\pi}{4}. \quad \square
\end{aligned}$$

e. Use the trigonometric identity $\tan^2(w) = \sec^2(w) - 1$:

$$\int \tan^2(w)\,dw = \int \left(\sec^2(w)-1\right)\,dw = \int \sec^2(w)\,dw - \int 1\,dw = \tan(w) - w + C \quad \square$$

f. Use the substitution $u = 2t$, so $du = 2\,dt$ and $\frac{1}{2}\,du = dt$. Note that $\begin{array}{ccc} x & 0 & \ln(2) \\ u & 0 & 2\ln(2) \end{array}$.

$$\begin{aligned}
\int_0^{\ln(2)} e^{2t}\,dt &= \int_0^{2\ln(2)} e^u \frac{1}{2}\,du = \frac{1}{2}\,e^u|_0^{2\ln(2)} = \frac{1}{2}\left(e^{2\ln(2)} - e^0\right) \\
&= \frac{\left(e^{\ln(2)}\right)^2 - 1}{2} = \frac{2^2-1}{2} = \frac{4-1}{2} = \frac{3}{2} \quad \square
\end{aligned}$$

3. Do any *five* (5) of **a**–**i**. *[25 = 5 × 5 each]*

a. Use the limit definition of the derivative to compute $g'(0)$ for $g(x) = 2x + 1$.

b. Determine whether the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\ln(n)}$ converges or diverges.

c. Find the Taylor series of $f(x) = e^{x+1}$ at $a = 0$.

d. Sketch the polar curve $r = 1$, where $0 \leq \theta \leq 2\pi$, and find the area of the region it encloses.

e. Sketch the surface obtained by rotating $y = \dfrac{x^2}{2}$, $0 \leq x \leq 2$, about the y-axis, and find its area.

f. Use the Right-hand Rule to compute the definite integral $\displaystyle\int_0^1 4x\,dx$.

g. Use the $\varepsilon - \delta$ definition of limits to verify that $\displaystyle\lim_{x \to 0} (2x - 1) = -1$.

h. Find the interval of convergence of the power series $\displaystyle\sum_{n=0}^{\infty} \frac{x^{2n}}{n!}$.

i. Sketch the solid obtained by rotating the region bounded by $y = x$, $y = 0$, and $x = 2$, about the x-axis, and find its volume.

Solutions to **3**.

a.
$$\begin{aligned} g'(0) &= \lim_{h \to 0} \frac{g(0+h) - g(0)}{h} = \lim_{h \to 0} \frac{g(h) - g(0)}{h} = \lim_{h \to 0} \frac{(2h-1) - (2 \cdot 0 - 1)}{h} \\ &= \lim_{h \to 0} \frac{2h - 1 - (-1)}{h} = \lim_{h \to 0} \frac{2h}{h} = \lim_{h \to 0} 2 = 2 \quad \square \end{aligned}$$

b. Note that $f(x) = \dfrac{1}{x\ln(x)}$ is positive and continuous for $x \geq 2$, and also decreasing, because $x\ln(x)$ is clearly increasing to ∞ as $x \to \infty$. Since $a_n = f(n)$, it follows from the Integral Test that the given series will converge or diverge depending on whether the improper integral $\displaystyle\int_2^{\infty} \frac{1}{x\ln(x)}\,dx$ converges or diverges. Since the improper integral actually diverges,

$$\int_2^{\infty} \frac{1}{x\ln(x)}\,dx = \lim_{t \to \infty} \int_2^t \frac{1}{x\ln(x)}\,dx = \lim_{t \to \infty} \int_{\ln(2)}^{\ln(t)} \frac{1}{u}\,du = \lim_{t \to \infty} \ln(u)\Big|_{\ln(2)}^{\ln(t)}$$

(Using the substitution $u = \ln(x)$, so $du = \dfrac{1}{x}\,dx$.)

$$= \lim_{t \to \infty} \left[\ln\left(\ln(t)\right) - \ln\left(\ln(2)\right)\right] = \infty \qquad \text{(As } \ln(z) \to \infty \text{ when } z \to \infty.\text{)}$$

the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\ln(n)}$ diverges as well. $\square$

c. *(Direct.)* Note that $f(x) = e^{x+1} = e \cdot e^x$, so $f'(x) = e \cdot \frac{d}{dx} e^x = e \cdot e^x = f(x)$. It follows that $f^{(n)}(x) = f(x) = e \cdot e^x$ for every $n \geq 0$, and so $f^{(n)}(0) = f(0) = e \cdot e^0 = e \cdot 1 = e$ for every $n \geq 0$. Plugging this into Taylor's Formula gives us

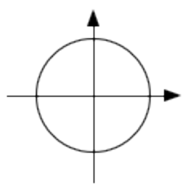
$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{e}{n!} x^n \quad \square$$

c. *(Indirect.)* We know that the Taylor series of e^x at 0 is $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$. Thus

$$f(x) = e^{x+1} = e \cdot e^x = e \sum_{n=0}^{\infty} \frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{e}{n!} x^n ,$$

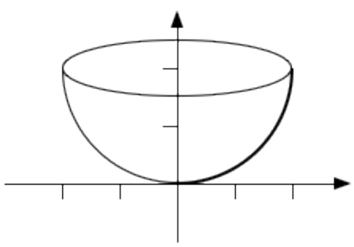
and this last must be the Taylor series for $f(x) = e^{x+1}$ at 0, because only one power series can fill this role. $\square$

d. Since $0 \leq \theta \leq 2\pi$, the curve goes all the way around the origin, and since $r = 1$ it follows that the curve consists of all the points which are a distance of 1 from the origin. This means that the curve is the circle of radius 1 centred at the origin – here's a sketch:



– and so has area $\pi 1^2 = \pi$. $\square$

e. Here's a sketch of the surface:



To find its area, we will use the formula for the area of a surface of revolution. Since

$y = \dfrac{x^2}{2}$, $\dfrac{dy}{dx} = \dfrac{1}{2}2x = x$. Hence

$$\text{Area} = \int_0^2 2\pi x \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \pi \int_0^2 2x\sqrt{1+x^2}\, dx = \pi \int_1^5 \sqrt{u}\, du$$

We substituted $u = 1 + x^2$, so $du = 2x\, dx$ and $\begin{matrix} x & 0 & 2 \\ u & 1 & 5 \end{matrix}$.

$$= \pi \int_1^5 u^{1/2}\, du = \frac{2}{3}\pi u^{3/2}\Big|_1^5 = \frac{2}{3}\pi 5\sqrt{5} - \frac{2}{3}\pi 1\sqrt{1} = \frac{2}{3}\pi\left(5\sqrt{5}-1\right). \quad \square$$

f. Plug and chug! Recall that the Right-hand Rule formula is

$$\int_a^b f(x)\, dx = \lim_{n\to\infty} \sum_{i=1}^n \frac{b-a}{n} \cdot f\left(a + i\frac{b-a}{n}\right),$$

so

$$\int_0^1 4x\, dx = \lim_{n\to\infty} \sum_{i=1}^n \frac{1-0}{n} \cdot 4\left(0 + i\frac{1-0}{n}\right) = \lim_{n\to\infty} \sum_{i=1}^n \frac{1}{n}\cdot 4\frac{i}{n}$$
$$= \lim_{n\to\infty} \frac{4}{n^2}\sum_{i=1}^n i = \lim_{n\to\infty} \frac{4}{n^2}\cdot\frac{n(n+1)}{2} = \lim_{n\to\infty} \frac{2(n+1)}{n}$$
$$= \lim_{n\to\infty}\left(\frac{2n}{n} + \frac{2}{n}\right) = \lim_{n\to\infty}\left(2 + \frac{2}{n}\right) = 2 + 0 = 2. \quad \square$$

g. We need to verify that for every $\varepsilon > 0$, there is some $\delta > 0$, such that if $|x - 0| < \delta$, then $|(2x-1) - (-1)| < \varepsilon$. As usual, we will try to reverse-engineer the necessary δ from ε. Suppose an $\varepsilon > 0$ is given. Then

$$|(2x-1)-(-1)| < \varepsilon \Leftrightarrow |2x - 1 + 1| < \varepsilon \Leftrightarrow |2x| < \varepsilon \Leftrightarrow |x| < \frac{\varepsilon}{2} \Leftrightarrow |x - 0| < \frac{\varepsilon}{2},$$

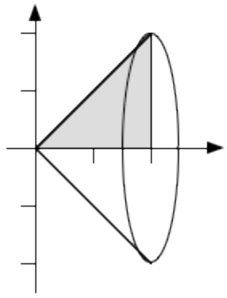
so $\delta = \dfrac{\varepsilon}{2}$ will do the job. Note that every step of our reverse-engineering process above is reversible, so if $|x - 0| < \delta = \dfrac{\varepsilon}{2}$, then $|(2x-1)-(-1)| < \varepsilon$. $\square$

h. We will use the Ratio Test to find the radius of convergence.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{x^{2(n+1)}}{(n+1)!}}{\frac{x^{2n}}{n!}}\right| = \lim_{n\to\infty}\left|\frac{x^{2n+2}}{x^{2n}}\cdot\frac{n!}{(n+1)!}\right| = \lim_{n\to\infty}\left|\frac{x^2}{n+1}\right| = 0,$$

since $x^2 \to x^2$ and $n+1 \to \infty$ as $n \to \infty$. It follows that $\displaystyle\sum_{n=0}^{\infty} \frac{x^{2n}}{n!}$ converges (absolutely) for all x, *i.e.* the radius of convergence is $R = \infty$ and the interval of convergence is $(-\infty, \infty)$. $\square$

i. Here's a sketch of the solid:



We will use the disk/washer method to find its volume. Note that the cross-section of the solid for a fixed value of x is a washer outer radius $R = y - 0 = y = x$ and inner radius $r = 0$ (so it's really a disk). Thus

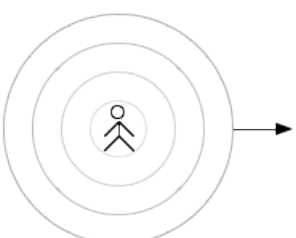
$$\text{Volume} = \int_0^2 \pi\left(R^2 - r^2\right)\,dx = \int_0^2 \pi\left(x^2 - 0^2\right)\,dx$$
$$= \int_0^2 \pi x^2\,dx = \pi\frac{x^3}{3}\Big|_0^2 = \pi\frac{2^3}{3} - \pi\frac{0^3}{3} = \frac{8}{3}\pi\,. \quad \square$$

Part Y. Do any *three* (3) of **4**–**7**. *[45 = 3 × 15 each]*

4. A zombie is dropped into a still pool, creating a circular ripple that moves outward from the point of impact at a constant speed. After 2 s the length of the ripple is increasing at a rate of $2\pi\ m/s$. How is the area enclosed by the ripple changing at this instant?

Hint: You have (just!) enough information to work out how the radius of the ripple changes with time.

SOLUTION. Recall that the length, *i.e.* circumference, of a circle of radius r is $C = 2\pi r$, while the area it encloses is $A = \pi r^2$. In this case $r = r(t)$ changes with time: what we are given amounts to (1) $\frac{dr}{dt}$ is constant and (2) $\left.\frac{dC}{dt}\right|_{t=2} = 2\pi$, and what we are being asked to compute is $\left.\frac{dA}{dt}\right|_{t=2}$.



Ignoring the hint for the moment, observe that $\frac{dC}{dt} = \frac{d}{dt}2\pi r = 2\pi\frac{dr}{dt}$. Then

$$\frac{dA}{dt} = \frac{d}{dt}\pi r^2 = \pi\frac{dr^2}{dr}\cdot\frac{dr}{dt} = \pi 2r\frac{dr}{dt} = r\left(2\pi\frac{dr}{dt}\right) = r\frac{dC}{dt}\,,$$

so $\left.\frac{dA}{dt}\right|_{t=2} = r(2)\left.\frac{dC}{dt}\right|_{t=2} = 2\pi r(2)$. The problem is that we do not know what $r(2)$ is.

Following the hint, we will try to figure out the radius of the ripple, $r(t)$, as a function of time, and then compute $r(2)$. There is one more piece of information in the set-up that we have not made explicit yet: (0) $r(0) = 0$, since the ripple was only created at the instant that the zombie fell into the pool.

Now, since $\frac{dr}{dt}$ is constant, $r(t) = mt + b$ for some constants m and b. (Obviously, $m = \frac{dr}{dt}$.) Note that $b = m0 + b = r(0) = 0$, using the observation above, so $r(t) = mt$. Since

$$2\pi = \left.\frac{dC}{dt}\right|_{t=2} = 2\pi\left.\frac{dr}{dt}\right|_{t=2} = 2\pi m\,,$$

we have that $m = \frac{dr}{dt} = 1$, so $r(t) = t$. It follows that $r(2) = 2$, and thus $\left.\frac{dA}{dt}\right|_{t=2} = r(2)\left.\frac{dC}{dt}\right|_{t=2} = 2\pi r(2) = 2\pi\cdot 2 = 4\pi$. $\square$

5. Find all the intercepts, maximum, minimum, and inflection points, and all the vertical and horizontal asymptotes of $h(x) = \frac{x}{1-x^2}$, and sketch its graph.

SOLUTION. *i. (y-intercept)* Plug in $x = 0$: $h(0) = \frac{0}{1-0^2} = \frac{0}{1} = 0$.

ii. (x-intercept) $y = h(x) = \frac{x}{1-x^2} = 0$ if and only if $x = 0$.

Note: Thus the sole x- and y-intercept is the point $(0, 0)$.

iii. (Vertical asymptotes) $h(x) = \frac{x}{1-x^2}$ is a rational function and hence defined and continuous whenever the denominator $1 - x^2 \neq 0$, so any vertical asymptotes could occur only when $1 - x^2 = 0$, *i.e.* when $x = \pm 1$. We'll take limits from both directions at $x = \pm 1$ and see what happens. Note that $1 - x^2 \to 0$ as $|x| \to 1$, so each of these limits must approach some infinity.

$$\lim_{x\to -1^-}\frac{x}{1-x^2} = +\infty \quad \text{since } 1-x^2 < 0 \text{ when } x < -1;$$
$$\lim_{x\to -1^+}\frac{x}{1-x^2} = -\infty \quad \text{since } 1-x^2 > 0 \text{ when } -1 < x < 0;$$
$$\lim_{x\to 1^-}\frac{x}{1-x^2} = +\infty \quad \text{since } 1-x^2 > 0 \text{ when } 0 < x < 1;$$
$$\lim_{x\to 1^+}\frac{x}{1-x^2} = -\infty \quad \text{since } 1-x^2 < 0 \text{ when } x > 1;$$

Thus $h(x)$ has vertical asymptotes at both $x = -1$ and $x = +1$, heading up to $+\infty$ from the left and down to $-\infty$ from the right of each point.

iv. (Horizontal asymptotes) One could compute the necessary limits with the help of l'Hôpital's Rule, but it's pretty easy to do with a little algebra too:

$$\lim_{x\to-\infty} \frac{x}{1-x^2} = \lim_{x\to-\infty} \frac{x}{1-x^2} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}} = \lim_{x\to-\infty} \frac{\frac{x}{x^2}}{\frac{1}{x^2} - \frac{x^2}{x^2}} = \lim_{x\to-\infty} \frac{\frac{1}{x}}{\frac{1}{x^2} - 1} = \frac{0}{0-1} = 0$$

$$\lim_{x\to+\infty} \frac{x}{1-x^2} = \lim_{x\to+\infty} \frac{x}{1-x^2} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}} = \lim_{x\to+\infty} \frac{\frac{x}{x^2}}{\frac{1}{x^2} - \frac{x^2}{x^2}} = \lim_{x\to+\infty} \frac{\frac{1}{x}}{\frac{1}{x^2} - 1} = \frac{0}{0-1} = 0$$

Thus $h(x)$ has the horizontal asymptote $y = 0$ in both directions.

v. (Maxima and minima) Note that $h(x)$ is defined and differentiable wherever it is defined because it is a rational function. First, we compute $h'(x)$ using the Quotient Rule:

$$h'(x) = \frac{d}{dx}\left(\frac{x}{1-x^2}\right) = \frac{\frac{dx}{dx} \cdot \left(1-x^2\right) - x \cdot \frac{d}{dx}\left(1-x^2\right)}{\left(1-x^2\right)^2}$$
$$= \frac{1 \cdot \left(1-x^2\right) - x \cdot (0-2x)}{\left(1-x^2\right)^2} = \frac{1-x^2+2x^2}{\left(1-x^2\right)^2} = \frac{1+x^2}{\left(1-x^2\right)^2}$$

Second, we find all the critical points, *i.e.* the points where $h'(x)$ is 0 or undefined. $h'(x) = \dfrac{1+x^2}{\left(1-x^2\right)^2} = 0$ exactly when $1+x^2 = 0$, which is to say never, since $1+x^2 \geq 1 > 0$ for all x. $h'(x) = \dfrac{1+x^2}{\left(1-x^2\right)^2}$ is defined and continuous unless $1-x^2 = 0$, *i.e.* when $x = \pm 1$. Since $h(x)$ has no critical points except where it has vertical asymptotes, $h(x)$ has no local maxima or minima.

Just for fun, here is the usual table for intervals of increase and decrease:

x	< -1	$\in (-1,1)$	> 1
$h'(x)$	$-$	$+$	$-$
$h(x)$	$\downarrow$	$\uparrow$	$\downarrow$

vi. (Inflection points) First, we compute $h''(x)$ using the Quotient Rule:

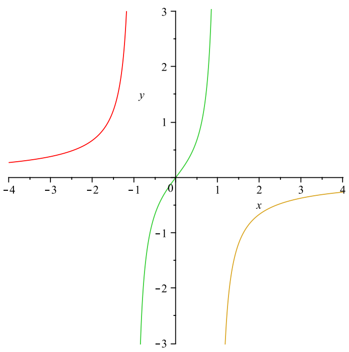
$$h''(x) = \frac{d}{dx}\left(\frac{1+x^2}{\left(1-x^2\right)^2}\right) = \frac{\frac{d}{dx}\left(1+x^2\right) \cdot \left(1-x^2\right)^2 - \left(1+x^2\right) \cdot \frac{d}{dx}\left(1-x^2\right)^2}{\left(1-x^2\right)^4}$$
$$= \frac{2x \cdot \left(1-x^2\right)^2 - \left(1+x^2\right) \cdot 2\left(1-x^2\right)(0-2x)}{\left(1-x^2\right)^4}$$
$$= \frac{2x \cdot \left(1-x^2\right) - \left(1+x^2\right) \cdot 2(-2x)}{\left(1-x^2\right)^3} = \frac{2x - 2x^3 + 4x + 4x^3}{\left(1-x^2\right)^3}$$
$$= \frac{6x + 2x^3}{\left(1-x^2\right)^3} = \frac{2x\left(3+x^2\right)}{\left(1-x^2\right)^3}$$

Second, we find all the potential inflection points, *i.e.* the points where $h''(x)$ is 0 or undefined. $h''(x) = \dfrac{2x\left(3+x^2\right)}{\left(1-x^2\right)^3} = 0$ exactly when either $x=0$ or when $3+x^2=0$ (which can't happen because $3+x^2 \geq 3 > 0$ for all x). $h''(x) = \dfrac{2x\left(3+x^2\right)}{\left(1-x^2\right)^3}$ is defined and continuous unless $1-x^2=0$, *i.e.* when $x=\pm 1$, which are the points where $h(x)$ has vertical asymptotes. Thus $x=0$ is the only possible point of inflection. Third, we test it to see if it really is, building the usual table:

x	< -1	$\in (-1,0)$	0	$\in (0,1)$	> 1
$h''(x)$	$+$	$-$	0	$+$	1
concavity	$\uparrow$	$\downarrow$		$\uparrow$	$\downarrow$

Thus $x=0$ is indeed an inflection point of $h(x)$.

vii. (Graph) Here is the graph of $h(x) = \frac{x}{1-x^2}$:



Cheating just a little, it was generated using the following `Maple` command:

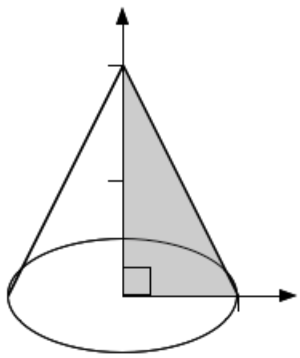
```
> plot([[s,s/(1-s^2),s=-4..-1],[s,s/(1-s^2),s=-1..1],
  [s,s/(1-s^2),s=1..4]],x=-4..4,y=-3..3);
```

□

6. Show that a cone with base radius 1 and height 2 has volume $\frac{2}{3}\pi$.

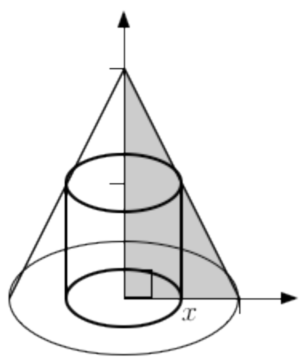
Hint: It's a solid of revolution ...

Solution. Following the hint, we need to realize the cone as a solid of revolution. One can obtain a ("right-circular") cone by rotating a right-angled triangle whose base is a radius of the base of the cone and whose altitude is the axis of symmetry of the cone. The hypotenuse of the triangle then runs along the surface of the cone from the tip to the base. We will set up our coordinate system so that the origin is at the right angle of the triangle, the x-axis runs along the base (the side that is the radius of the cone), and the y-axis runs along the altitude (the side that is the axis of symmetry of the cone). Since the cone has



base radius 1 and height 2, the hypotenuse of the triangle is the line segment joining $(0, 2)$ to $(1, 0)$.as in the diagram. It is not hard to work out that the equation of the line forming the hypotenuse of the triangle is $y = -2x + 2$.

Thus we are trying to find the volume of the solid of revolution obtained by revolving the region bounded by $x = 0$, $y = 0$, and $y = -2x + 2$ about the y-axis, *i.e.* about the line $x = 0$. The volume can be computed readily using either the disk/washer method or the method of cylindrical shells. Since we used disks in the solution to **3i** (note that the solid of revolution there was a cone as well), we will use cylindrical shells here.



Since we revolved the region about the y-axis, a typical shell is obtained by revolving a vertical cross-section of the region, that is, one for a fixed value of x, about the y-axis. The shell for a given value of x has radius $r = x$ and height $h = y - 0 = -2x + 2$, and so has surface area $2\pi rh = 2\pi x(-2x + 2) = 4\pi\left(x - x^2\right)$. Note that since we revolved the region about the y-axis and are using shells, we need to integrate with respect to x, and the possible values of x in the triangle are $0 \le x \le 1$. Thus the volume of the cone is:

$$\begin{aligned}
\text{Volume} &= \int_0^1 2\pi rh\,dx = \int_0^1 4\pi\left(x - x^2\right)\,dx = 4\pi\left(\frac{x^2}{2} - \frac{x^3}{3}\right)\Bigg|_0^1 \\
&= 4\pi\left[\left(\frac{1^2}{2} - \frac{1^3}{3}\right) - \left(\frac{0^2}{2} - \frac{0^3}{3}\right)\right] = 4\pi\left(\left[\frac{1}{2} - \frac{1}{3}\right] - 0\right) \\
&= 4\pi\frac{1}{6} = \frac{2}{3}\pi \quad \square
\end{aligned}$$

7. Do all *four* (4) of **a**–**d**.

a. Use Taylor's formula to find the Taylor series of $f(x) = \sin(x)$ at $a = 0$. *[7]*

b. Determine the radius and interval of convergence of this Taylor series. *[4]*

c. Find the Taylor series of $g(x) = x\sin(x)$ at $a = 0$ by multiplying the Taylor series for $f(x) = \sin(x)$ by x. *[1]*

d. Use Taylor's formula and your series from **c** to compute $g^{(16)}(0)$. *[3]*

SOLUTION TO **a**. Recall that the Taylor series of $f(x)$ at $a = 0$ is $\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$, so we'll need to know what the values of the nth derivatives of $f(x) = \sin(x)$ are at 0:

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\sin(x)$	0
1	$\cos(x)$	1
2	$-\sin(x)$	0
3	$-\cos(x)$	-1
4	$\sin(x)$	0
5	$\cos(x)$	1
6	$-\sin(x)$	0
$\vdots$	$\vdots$	$\vdots$
$2k$	$(-1)^k \sin(x)$	0
$2k+1$	$(-1)^k \cos(x)$	$(-1)^k$
$\vdots$	$\vdots$	$\vdots$

Note that $f^{(n)}(0) = 0$ when n is even, and alternates between 1 and -1 when n is odd. It follows that the Taylor series of $f(x) = \sin(x)$ at $a = 0$ is $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$. $\square$

SOLUTION TO **b**. We will use the Ratio Test to find the radius of convergence.

$$\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} \left| \frac{\frac{(-1)^{k+1}}{(2(k+1)+1)!} x^{2(k+1)+1}}{\frac{(-1)^k}{(2k+1)!} x^{2k+1}} \right| = \lim_{k\to\infty} \left| \frac{(-1)^{k+1} x^{2k+3}}{(2k+3)!} \cdot \frac{(2k+1)!}{(-1)^k x^{2k+1}} \right|$$
$$= \lim_{k\to\infty} \left| \frac{(-1)x^2}{(2k+3)(2k+2)} \right| = \lim_{k\to\infty} \frac{x^2}{(2k+3)(2k+2)} = 0\,,$$

since $x^2 \to x^2$ and $(2k+3)(2k+2) \to \infty$ as $k \to \infty$. It follows that the series converges (absolutely) for all x, *i.e.* the radius of convergence is $R = \infty$ and the interval of convergence is $(-\infty, \infty)$. $\square$

SOLUTION TO **c**. If $g(x) = x\sin(x)$ then the Taylor series at $a = 0$ of $g(x)$ should be the Taylor series for $f(x) = \sin(x)$ multiplied by by x:

$$\sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!} x^n = x \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = x \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+2} \quad \square$$

SOLUTION TO **d**. By Taylor's formula the coefficient of x^{16} in the Taylor series of $g(x)$ is $\dfrac{g^{(16)}(0)}{16!}$. x^{16} occurs in $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+2}$ when $2k+2 = 16$, *i.e.* when $k = 7$. The coefficient of x^{16} is therefore $\dfrac{(-1)^7}{(2 \cdot 7 + 1)!} = \dfrac{-1}{15!}$. We compare this to the coefficient given by Taylor's formula, $\dfrac{g^{(16)}(0)}{16!} = \dfrac{-1}{15!}$, and solve for $g^{(16)}(0) = 16! \dfrac{-1}{15!} = -16$. □

[Total = 100]

Part Z. Bonus problems! Do them (or not), if you feel like it.

$\ln\left(\frac{1}{e}\right)$**.** Does $\displaystyle\lim_{n\to\infty} \left[\left(\sum_{k=1}^{n} \frac{1}{k} \right) - \ln(n) \right]$ exist? Explain why or why not. *[2]*

SOLUTION. It does exist. The real number computed by this limit is often called the Euler-Mascheroni constant, about which surprisingly little is known. (For example, we don't even know if it is rational or irrational.) The Wikipedia article about it at

`http://en.wikipedia.org/wiki/Euler-mascheroni_constant`

is good place to start learning about it. A little searching (not in the article just referred to, although the graph there interpreting the limit as an area is a clue) should turn up a proof the limit exists. □

$\ln\left(\frac{1}{1}\right)$**.** Write a haiku touching on calculus or mathematics in general. *[2]*

haiku?

seventeen in three:
five and seven and five of
syllables in lines

SOLUTION. You're on your own on this one! □