Math 1100 — Calculus, Quiz #8B — 2009-12-03

(0) 1. Compute $\lim_{x\to 0}\left(\cot(x) - \frac{1}{x}\right)$. ($\textit{Hint:}$ $\cot(x) = \frac{\cos(x)}{\sin(x)}$)

Note: *There was an typo in the 'hint' for this question, which no doubt caused a great deal of confusion during the quiz. Therefore, this question is voided. Your mark is based only on question #2. But here is the solution to the question anyways (for whatever its worth).*

Solution: We have $\lim_{x\to 0}\cot(x) = \infty = \lim_{x\to 0}\frac{1}{x}$. Thus, this limit is an indeterminate form of type $(\infty - \infty)$. To apply l'Hospital's rule, we must put the two terms over a common denominator:

$$\cot(x) - \frac{1}{x} \quad = \quad \frac{\cos(x)}{\sin(x)} - \frac{1}{x} \quad = \quad \frac{x\,\cos(x) - \sin(x)}{x\,\sin(x)}. \tag{1}$$

Observe that $\lim_{x\to 0}(x\,\cos(x) - \sin(x)) = 0\cdot 1 - 0 = 0 = \lim_{x\to 0} x\,\sin(x)$. Thus, the limit of expression (1) is an indeterminate form of type $0/0$, so we can apply l'Hospital's rule. We have:

$$\lim_{x\to 0}\left(\cot(x) - \frac{1}{x}\right) \underset{(*)}{=\!=} \overbrace{\lim_{x\to 0}\frac{x\cos(x) - \sin(x)}{x\sin(x)}}^{\text{Indeterminate form of type } 0/0} \underset{\text{(H1)}}{=\!=} \overbrace{\lim_{x\to 0}\frac{-x\sin(x)}{\sin(x) + x\cos(x)}}^{\text{Indeterminate form of type } 0/0}$$

$$\underset{\text{(H2)}}{=\!=} \lim_{x\to 0}\frac{-\sin(x) - x\cos(x)}{2\cos(x) - x\sin(x)} = \frac{\lim_{x\to 0}(-\sin(x) - x\cos(x))}{\lim_{x\to 0}(2\cos(x) - x\sin(x))}$$

$$= \frac{-\sin(0) - 0\cos(0)}{2\cos(0) - 0\sin(0)} = \frac{0}{2} = \boxed{0.}$$

Here, $(*)$ is by equation (1). Next, (H1) is l'Hospital's rule with $f(x) := x\cos(x) - \sin(x)$ so that $f'(x) = \cos(x) - x\sin(x) - \cos(x) = -x\sin(x)$, while $g(x) := x\sin(x)$, so that $g'(x) = \sin(x) + x\cos(x)$. Finally, (H2) is l'Hospital's rule with $f''(x) = -\sin(x) - x\cos(x)$ and $g''(x) = \cos(x) + \cos(x) - x\sin(x) = 2\cos(x) - x\sin(x)$. □

(100) 2. Compute $\lim_{x\searrow 0} x^{(x^2)}$.

Solution: We have $\lim_{x\searrow 0} x = 0 = \lim_{x\searrow 0} x^2$, so this limit is an indeterminate form of type 0^0. We take the logarithm and apply l'Hospital's rule. We have:

$$\ln\left(x^{(x^2)}\right) = x^2\cdot\ln(x). \qquad \text{Thus,}$$

$$\ln\left(\lim_{x\searrow 0} x^{(x^2)}\right) = \lim_{x\searrow 0}\ln\left(x^{(x^2)}\right) = \lim_{x\searrow 0} x^2\cdot\ln(x) = \lim_{x\searrow 0}\frac{\ln(x)}{x^{-2}} \qquad \text{(indet. form of type } \tfrac{\infty}{\infty})$$

$$\underset{\text{(H)}}{=\!=} \lim_{x\searrow 0}\frac{x^{-1}}{-2x^{-3}} = \lim_{x\searrow 0} -\frac{1}{2}x^{3-1} = -\frac{1}{2}\lim_{x\searrow 0} x^2 = 0.$$

Here, (H) is l'Hospital's rule, with $f(x) = \ln(x)$ so that $f'(x) = x^{-1}$, and $g(x) = x^{-2}$ so that $g'(x) = -2x^{-3}$.

This, of course, is the limit of the *logarithm* of the original expression. It follows that $\lim_{x \searrow 0} x^{(x^2)} = \exp(0) = \boxed{1.}$ □